chapter three
Rotation of Rigid Bodies

So far in this book we have considered the equations of motion and other dynamical formulae associated with a particle. In practice, however, an object is made of millions of particles, each at different places, and we need now to consider the motion of moving objects.

Moment of Inertia, I

Suppose a rigid object is rotating about a fixed axis O, and a particle A of the object makes an angle θ with a fixed line OY in space at some instant, Fig. 3.1. The angular velocity, $d\theta/dt$ or ω, of every particle about O is the same, since we are dealing with a rigid body, and the velocity v_1 of A at this instant is given by $r_1\omega$, where r_1 = OA. Thus the kinetic energy of A $= \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2$. Similarly, the kinetic energy of another particle of the body $= \frac{1}{2}m_2r_2^2\omega^2$, where r_2 is its distance from O and m_2 is its mass. In this way we see that the kinetic energy, K.E., of the whole object is given by

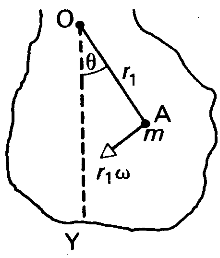

FIG. 3.1
Rotating rigid body

$$\text{K.E.} = \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2\omega^2 + \tfrac{1}{2}m_3r_3^2\omega^2 + \ldots$$
$$= \tfrac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots)$$
$$= \tfrac{1}{2}\omega^2(\Sigma mr^2),$$

where Σmr^2 represents the sum of the magnitudes of 'mr^2' for all the particles of the object. We shall see shortly how the quantity Σmr^2 can be calculated for a particular object. The magnitude of Σmr^2 is known as the *moment of inertia* of the object about the axis concerned, and we shall denote it by the symbol I. Thus

$$\text{Kinetic energy, K.E.,} = \tfrac{1}{2}I\omega^2. \qquad (1)$$

The units of I are *kg metre*^{2} (kg m^2). The unit of ω is 'radian s^{-1}' (rad s^{-1}). Thus if $I = 2$ kg m^2 and $\omega = 3$ rad s^{-1}, then

$$\text{K.E.} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\times 2\times 3^2 \text{ joule} = 9 \text{ J}.$$

The kinetic energy of a particle of mass m moving with a velocity v is $\frac{1}{2}mv^2$. It will thus be noted that the formula for the kinetic energy of a rotating object is similar to that of a moving particle, the mass m being replaced by the moment of inertia I and the velocity v being replaced by the angular velocity ω. As we shall require values of I, the moment of inertia of several objects about a particular axis will first be calculated.

Moment of Inertia of Uniform Rod

(1) *About axis through middle.* The moment of inertia of a small element δx about an axis PQ through its centre O perpendicular to the length $= \left(\frac{\delta x}{l}M\right)x^2$, where l is the length of the rod, M is its mass, and x is the distance of the small element from O, Fig. 3.2.

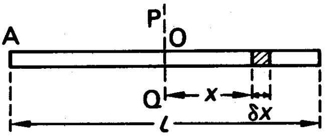

FIG. 3.2 Moment of inertia—uniform rod

$$\therefore \text{ moment of inertia, } I, = 2\int_o^{l/2}\left(\frac{dx}{l}M\right)x^2$$

$$= \frac{2M}{l}\int_o^{l/2} x^2 dx = \frac{Ml^2}{12} \quad . \quad (1)$$

Thus if the mass of the rod is 60 g and its length is 20 cm, $M = 6\times10^{-2}$ kg, $l = 0{\cdot}2$ m, and $I = 6\times10^{-2}\times0{\cdot}2^2/12 = 2\times10^{-4}$ kg m^2.

(2) *About the axis through one end, A.* In this case, measuring distances x from A instead of O,

$$\text{moment of inertia, } I, = \int_o^l\left(\frac{dx}{l}M\right)\times x^2 = \frac{Ml^2}{3} \quad . \quad (2)$$

Moment of Inertia of Ring

Every element of the ring is the same distance from the centre. Hence the moment of inertia about an axis through the centre perpendicular to the plane of the ring $= Ma^2$, where M is the mass of the ring and a is its radius.

Moment of Inertia of Circular Disc

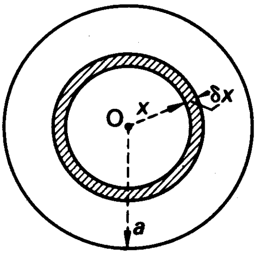

FIG. 3.3
Moment of inertia—disc

Consider the moment of inertia of a circular disc about an axis through its centre perpendicular to its plane, Fig. 3.3. If we take a small ring of the disc enclosed between radii x and $x+\delta x$, its mass $= \frac{2\pi x\delta x}{\pi a^2}M$, where a is the radius of the disc and M is its mass. Each element of the ring is distant x from the centre, and hence the moment of inertia of the ring about the axis through O $= \left(\frac{2\pi x\delta x}{\pi a^2}M\right)\times x^2$

$$\therefore \text{ moment of inertia of whole disc} = \int_0^a \frac{2\pi x dx}{\pi a^2} M \times x^2$$

$$= \frac{Ma^2}{2} \quad . \quad . \quad (1)$$

Thus if the disc weighs 60 g and has a radius of 10 cm, $M = 60$ g $= 6 \times 10^{-2}$ kg, $a = 0{\cdot}1$ m, so that $I = 6 \times 10^{-2} \times 0{\cdot}1^2/2 = 3 \times 10^{-4}$ kg m^2.

Moment of Inertia of Cylinder

If a cylinder is *solid*, its moment of inertia about the axis of symmetry is the sum of the moments of inertia of discs into which we may imagine the cylinder cut. The moment of inertia of each disc $= \frac{1}{2}$ mass $\times a^2$, where a is the radius; and hence, if M is the mass of the cylinder,

$$\text{moment of inertia of solid cylinder} = \tfrac{1}{2}Ma^2 \quad . \quad (i)$$

If a cylinder is *hollow*, its moment of inertia about the axis of symmetry is the sum of the moments of inertia of the curved surface and that of the two ends, assuming the cylinder is closed at both ends. Suppose a is the radius, h is the height of the cylinder, and σ is the mass per unit area of the surface. Then

$$\text{mass of curved surface} = 2\pi a h \sigma,$$

and $\quad$ moment of inertia about axis $=$ mass $\times a^2 = 2\pi a^3 h\sigma$,

since we can imagine the surface cut into rings.

The moment of inertia of one end of the cylinder $=$ mass $\times a^2/2 = \pi a^2 \sigma \times a^2/2 = \pi a^4 \sigma/2$. Hence the moment of inertia of both ends $= \pi a^4 \sigma$.

$$\therefore \text{ moment inertia of cylinder, } I, = 2\pi a^3 h\sigma + \pi a^4 \sigma.$$

The mass of the cylinder, $M, = 2\pi a h\sigma + 2\pi a^2 \sigma$

$$\therefore I = \frac{2\pi a^3 h\sigma + \pi a^4 \sigma}{2\pi a h\sigma + 2\pi a^2 \sigma} M.$$

$$= \frac{2a^2 h + a^3}{2h + 2a} M.$$

$$= \tfrac{1}{2}Ma^2 + \frac{a^2 h}{2h + 2a} M \quad . \quad . \quad . \quad (ii)$$

If a hollow and a solid cylinder have the same mass M and the same radius and height, it can be seen from (i) and (ii) that the moment of inertia of the hollow cylinder is greater than that of the solid cylinder about the axis of symmetry. This is because the mass is distributed on the average at a greater distance from the axis in the former case.

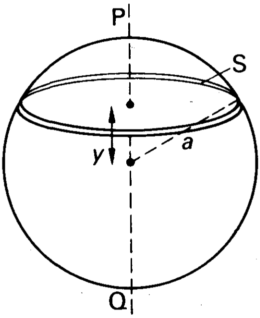

FIG. 3.4
Moment of inertia—sphere

Moment of Inertia of Sphere

The moment of inertia of a sphere about an axis PQ through its centre can be found by cutting thin discs such as S perpendicular to the axis, Fig. 3.4. The volume of the disc, of thickness δy and distance y from the centre,

$$= \pi r^2 \delta y = \pi(a^2 - y^2)\delta y.$$

$$\therefore \text{ mass } M' \text{ of disc} = \frac{\pi(a^2-y^2)\delta y}{4\pi a^3/3} M$$

$$= \frac{3M}{4a^3}(a^2-y^2)\delta y,$$

where M is the mass of the sphere and a is its radius, since the volume of the sphere $= 4\pi a^3/3$. Now the moment of inertia of the disc about PQ

$$= M' \times \frac{\text{radius}^2}{2} = \frac{3M}{4a^3}(a^2-y^2)\delta y \times \frac{(a^2-y^2)}{2}$$

$$\therefore \text{ moment of inertia of sphere} = \frac{3M}{8a^3}\int_{-a}^{+a}(a^4 - 2a^2y^2 + y^4)dy$$

$$= \frac{2}{5}Ma^2 \quad . \quad . \quad . \quad (1)$$

Thus if the sphere has mass 4 kg and a radius of 0·2 m, the moment of inertia $= \frac{2}{5} \times 4 \times 0{\cdot}2^2 = 0{\cdot}064$ kg m^2.

Radius of Gyration

The moment of inertia of an object about an axis, Σmr^2, is sometimes written as Mk^2, where M is the mass of the object and k is a quantity called the *radius of gyration* about the axis. For example, the moment of inertia of a rod about an axis through one end $= Ml^2/3$ (p. 76) $= M(l/\sqrt{3})^2$. Thus the radius of gyration, k, $= l/\sqrt{3} = 0{\cdot}58l$. The moment of inertia of a sphere about its centre $= \frac{2}{5}Ma^2 = M \times (\sqrt{\frac{2}{5}}a)^2$. Thus the radius of gyration, k, $= \sqrt{\frac{2}{5}}a = 0{\cdot}63a$ in this case.

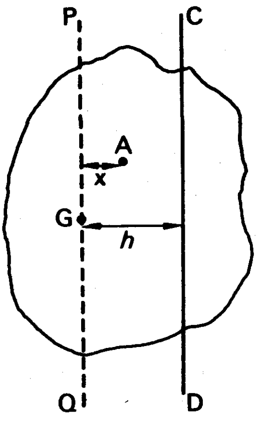

FIG. 3.5
Theorem of parallel axes

Relation Between Moment of Inertia About C.G. and Parallel Axis.

Suppose I is the moment of inertia of a body about an axis CD and I_G is the moment of inertia about a parallel axis PQ through the centre of gravity, G, distant h from the axis CD, Fig. 3.5. If A is a particle of mass m whose distance from PQ is x, its moment of inertia about CD $= m(h-x)^2$

$$\therefore I = \Sigma m(h-x)^2 = \Sigma mh^2 + \Sigma mx^2 - \Sigma 2mhx.$$

Now $\Sigma mh^2 = h^2 \times \Sigma m = Mh^2$, where M is the total mass of the object, and $\Sigma mx^2 = I_G$, the moment of inertia through the centre of gravity.

Also, $$\Sigma 2mhx = 2h\Sigma mx = 0,$$

since Σmx, the sum of the moments about the centre of gravity, is zero; this follows because the moment of the resultant (the weight) about G is zero.

$$\therefore I = I_G + Mh^2 \quad . \quad . \quad . \quad . \quad (1)$$

From this result, it follows that the moment of inertia, I, of a disc of radius a and mass M about an axis through a point on its circumference $= I_G + Ma^2$, since $h = a =$ radius of disc in this case. But $I_G =$ moment of inertia about the centre $= Ma^2/2$ (p.77).

$$\therefore \text{moment of inertia, } I, = \frac{Ma^2}{2} + Ma^2 = \frac{3Ma^2}{2}.$$

Similarly the moment of inertia of a sphere of radius a and mass M about an axis through a point on its circumference $= I_G + Ma^2 = 2Ma^2/5 + Ma^2 = 7Ma^2/5$, since I_G, the moment of inertia about an axis through its centre, is $2Ma^2/5$.

Relation Between Moments of Inertia about Perpendicular Axes

Suppose OX, OY are any two perpendicular axes and OZ is an axis perpendicular to OX and OY, Fig. 3.6 (i). The moment of inertia, I, of

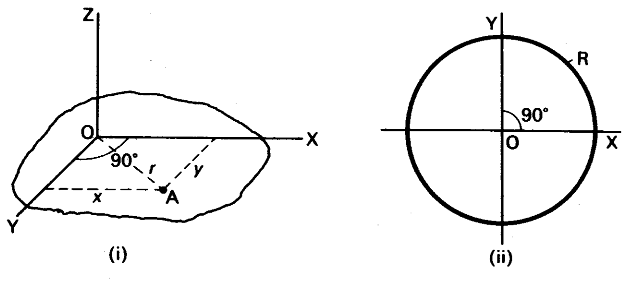

FIG. 3.6 Theorem of perpendicular axes

a body about the axis OZ $= \Sigma mr^2$, where r is the distance of a particle A from OZ and m is its mass. But $r^2 = x^2 + y^2$, where x, y are the distances of A from the axis OY, OX respectively.

$$\therefore I = \Sigma m(x^2 + y^2) = \Sigma mx^2 + \Sigma my^2.$$

$$\therefore I = I_y + I_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where I_y, I_x are the moments of inertia about OX, OY respectively.

As a simple application, consider a ring R and two perpendicular axes OX, OY in its plane, Fig. 3.6 (ii). Then from the above result,

$I_y + I_x = I =$ moment of inertia through O perpendicular to ring.

$$\therefore I_y + I_x = Ma^2.$$

But $I_y = I_x$, by symmetry.

$$\therefore I_x + I_x = Ma^2,$$

$$\therefore I_x = \frac{Ma^2}{2}.$$

This is the moment of inertia of the ring about any diameter in its plane.

In the same way, the moment of inertia, I, of a *disc* about a diameter in its plane is given by

$$I + I = \frac{Ma^2}{2},$$

since the moments of inertia, I, about the two perpendicular diameters are the same and $Ma^2/2$ is the moment of inertia of the disc about an axis perpendicular to its plane.

$$\therefore I = \frac{Ma^2}{4}.$$

Couple on a Rigid Body

Consider a rigid body rotating about a fixed axis O with an angular velocity ω at some instant. Fig. 3.7.

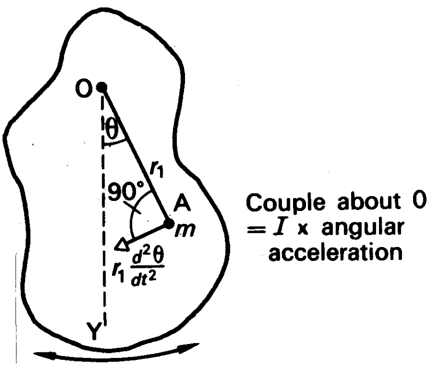

FIG. 3.7 Couple on rigid body

The force acting on the particle A $= m_1 \times$ acceleration $= m_1 \times \frac{d}{dt}(r_1\omega) = m_1 \times r_1\frac{d\omega}{dt} = m_1 r_1 \frac{d^2\theta}{dt^2}$, since $\omega = \frac{d\theta}{dt}$. The moment of this force about the axis O = force × perpendicular distance from O = $m_1 r_1 \frac{d^2\theta}{dt^2} \times r_1$, since the force acts perpendicularly to the line OA.

$$\therefore \text{ moment or } torque = m_1 r_1{}^2 \frac{d^2\theta}{dt^2}.$$

∴ total moment of all forces on body about O, or *torque*,

$$= m_1 r_1{}^2 \frac{d^2\theta}{dt^2} + m_2 r_2{}^2 \frac{d^2\theta}{dt^2} + m_3 r_3{}^2 \frac{d^2\theta}{dt^2} + \ldots$$

$$= (\Sigma mr^2) \times \frac{d^2\theta}{dt^2},$$

since the angular acceleration, $d^2\theta/dt^2$, about O is the same for all particles.

$$\therefore \text{ total torque about O } = I \frac{d^2\theta}{dt^2}, \quad \quad (1)$$

where $I = \Sigma mr^2$ = moment of inertia about O. The moment about O is produced by external forces which together act as a *couple* of torque C say. Thus, for any rotating rigid body,

$$\textit{Couple, } C = I \frac{d^2\theta}{dt^2}.$$

This result is analogous to the case of a particle of mass m which undergoes an acceleration a when a force F acts on it. Here $F = ma$. In place of F we have a *couple* C for a rotating rigid object; in place of m we have the *moment of inertia* I; and in place of linear acceleration a, we have *angular acceleration* $d^2\theta/dt^2 (d\omega/dt)$.

EXAMPLES

1. A heavy flywheel of mass 15 kg and radius 20 cm is mounted on a horizontal axle of radius 1 cm and negligible mass compared with the flywheel. Neglecting friction, find (i) the angular acceleration if a force of 4 kgf is applied tangentially to the axle, (ii) the angular velocity of the flywheel after 10 seconds.

(i) $$\text{Moment of inertia} = \frac{Ma^2}{2} = \frac{15 \times 0{\cdot}2^2}{2} = 0{\cdot}3 \text{ kg m}^2.$$

$$\text{Couple } C = 4 \times 9{\cdot}8 \text{ (N)} \times 0{\cdot}01 \text{ (m)} = 0{\cdot}4 \text{ N m approx.}$$

$$\therefore \text{ angular acceleration} = \frac{0{\cdot}4}{0{\cdot}3} = 1{\cdot}3 \text{ rad s}^{-2}.$$

(ii) After 10 seconds, angular velocity = angular acceleration × time.

$$= 1{\cdot}3 \times 10 = 13 \text{ rad s}^{-1}.$$

2. The moment of inertia of a solid flywheel about its axis is 0·1 kg m². It is set in rotation by applying a tangential force of 2 kgf with a rope wound round the circumference, the radius of the wheel being 10 cm. Calculate the angular acceleration of the flywheel. What would be the acceleration if a mass of 2 kg were hung from the end of the rope? (*O. & C.*)

Couple $C = I\frac{d^2\theta}{dt^2}$ = moment of inertia × angular acceleration.

Now $C = 2 \times 9{\cdot}8 \times 0{\cdot}1$ N m.

$$\therefore \text{angular acceleration} = \frac{2 \times 9{\cdot}8 \times 0{\cdot}1}{0{\cdot}1}$$

$$= 19{\cdot}6 \text{ rad s}^{-2}.$$

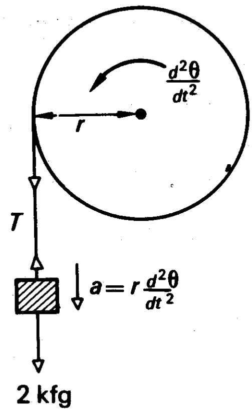

FIG. 3.8 Example

If a mass of 2kg is hung from the end of the rope, it moves down with an acceleration a. Fig. 3.8. In this case, if T is the tension in the rope,

$$mg - T = ma \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the flywheel,
$$T.r = \text{couple} = I\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where r is the radius of the flywheel. Now the mass of 2 kg descends a distance given by $r\theta$, where θ is the angle the flywheel has turned. Hence the acceleration $a = rd^2\theta/dt^2$. Substituting in (1),

$$\therefore mg - T = mr\frac{d^2\theta}{dt^2}.$$

$$\therefore mgr - T.r = mr^2\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Adding (2) and (3),

$$\therefore mgr = (I + mr^2)\frac{d^2\theta}{dt^2}.$$

$$\therefore \frac{d^2\theta}{dt^2} = \frac{mgr}{I + mr^2} = \frac{2 \times 10 \times 0{\cdot}1}{0{\cdot}1 + 2 \times 0{\cdot}1^2}.$$

$$= 16{\cdot}7 \text{ rad s}^{-2}.$$

using $g = 10 \text{ m s}^{-2}$.

Angular Momentum and Conservation

In linear or straight-line motion, an important property of a moving object is its linear momentum (p. 18). When an object spins or rotates about an axis, its *angular momentum* plays an important part in its motion.

Consider a particle A of a rigid object rotating about an axis O. Fig. 3.9. The momentum of A = mass × velocity = $m_1 v = m_1 r_1 \omega$. The 'angular momentum' of A about O is defined as the *moment of the momentum* about O. Its magnitude is thus $m_1 v \times p$, where p is the perpendicular distance from O to the direction of v. Thus angular momentum of A = $m_1 vp = m_1 r_1 \omega \times r_1 = m_1 {r_1}^2 \omega$.

$\therefore$ total angular momentum of whole body $= \Sigma m_1 r_1{}^2 \omega = \omega \Sigma m_1 r_1{}^2$
$= I\omega,$

where I is the moment of inertia of the body about O.

Angular momentum is analogous to 'linear momentum', mv, in the dynamics of a moving particle. In place of m we have I, the moment of inertia; in place of v we have ω, the angular velocity.

Further, the *conservation of angular momentum*, which corresponds to the conservation of linear momentum, states that *the angular momentum about an axis of a given rotating body or system of bodies is constant, if no external couple acts about that axis.* Thus when a high diver jumps from a diving board, his moment of inertia, I, can be decreased by curling his body more, in which case his angular velocity ω is increased. Fig. 3.9 (ii). He may then be able to turn more somersaults before striking the water. Similarly, a dancer on skates can spin faster by folding her arms.

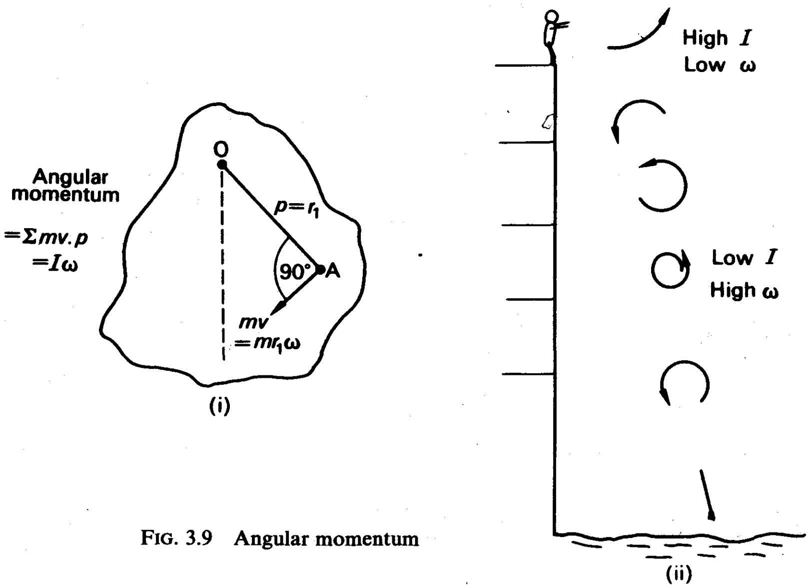

FIG. 3.9 Angular momentum

The earth is an object which rotates about an axis passing through its geographic north and south poles with a period of 1 day. If it is struck by meteorites, then, since action and reaction are equal, no external couple acts on the earth and meteorites. Their total angular momentum is thus conserved. Neglecting the angular momentum of the meteorites about the earth's axis before collision compared with that of the earth, then

angular momentum of *earth plus meteorites* after collision =
angular momentum of *earth* before collision.

Since the effective mass of the earth has increased after collision the moment of inertia has increased. Hence the earth will slow up slightly.

Similarly, if a mass is dropped gently on to a turntable rotating freely at a steady speed, the conservation of angular momentum leads to a reduction in the speed of the table.

Angular momentum, and the principle of the conservation of angular momentum, have wide applications in physics. They are used in connection with enormous rotating masses such as the earth, as well as minute spinning particles such as electrons, neutrons and protons found inside atoms.

Experiment on Conservation of Angular Momentum

A simple experiment on the principle of the conservation of angular momentum is illustrated below.

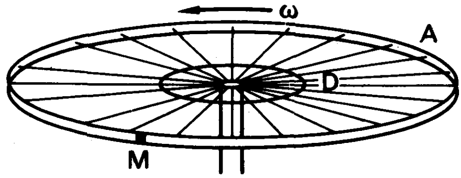

FIG. 3.10 Conservation of angular momentum

Briefly, in Fig. 3.10 (i) a bicycle wheel A without a tyre is set rotating in a horizontal plane and the time for three complete revolutions is obtained with the aid of a white tape marker M on the rim. A ring D of known moment of inertia, I, is then gently placed on the wheel concentric with it, by 'dropping' it from a small height. The time for the next three revolutions is then determined. This is repeated with several more rings of greater known moment of inertia.

If the principle of conservation of angular momentum is true, then $I_0\omega_0 = (I_0+I_1)\omega_1$, where I_0 is the moment of inertia of the wheel alone, ω_0 is the angular frequency of the wheel alone, and ω_1 is the angular frequency with a ring. Thus is t_0, t_1 are the respective times for three revolutions,

$$\frac{I_0+I_1}{t_1} = \frac{I_0}{t_0}$$

$$\therefore \frac{I_1}{I_0}+1 = \frac{t_1}{t_0}.$$

Thus a graph of t_1/t_0 v. I_1 should be a straight line. Within the limits of experimental error, this is found to be the case.

EXAMPLE

Consider a disc of mass 100 g and radius 10 cm is rotating freely about axis O through its centre at 40 r.p.m. Fig. 3.11. Then, about O,

$$\text{moment of inertia } I = \frac{Ma^2}{2} = \tfrac{1}{2}\times 0{\cdot}1\ (\text{kg})\times 0{\cdot}1^2\ (\text{m}^2) = 5\times 10^{-4}\ \text{kg m}^2,$$

and angular momentum $= I\omega = 5\times 10^{-4}\omega$,
where ω is the angular velocity corresponding to 40 r.p.m.

Suppose some wax W of mass m 20 g is dropped gently on to the disc at a distance r of 8 cm from the centre O. The disc then slows down to another speed, corresponding to an angular velocity ω_1 say. The total angular momentum about O of disc plus wax

$$= I\omega_1 + mr^2\omega_1 = 5\times10^{-4}\omega_1 + 0{\cdot}02\times0{\cdot}08^2 . \omega_1$$
$$= 6{\cdot}28\times10^{-4}\omega_1.$$

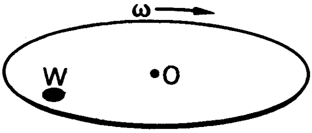

FIG. 3.11 Example

From the conservation of angular momentum for the disc and wax about O

$$6{\cdot}28\times10^{-4}\omega_1 = 5\times10^{-4}\omega.$$
$$\therefore \frac{\omega_1}{\omega} = \frac{500}{628} = \frac{n}{40},$$

where n is the r.p.m. of the disc.

$$\therefore n = \frac{500}{628}\times40 = 32 \text{ (approx).}$$

Kepler's law and angular momentum

Consider a planet moving in an orbit round the sun S. Fig. 3.12.

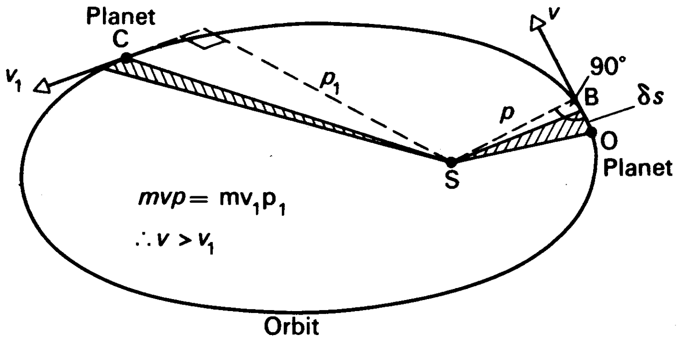

FIG. 3.12 Angular momentum and planets

At an instant when the planet is at O, its velocity v is along the tangent to the orbit at O. Suppose the planet moves a very small distance δs from O to B in a small time δt, so that the velocity $v = \delta s/\delta t$ and its direction is practically along OB. Then, if the conservation of angular momentum is obeyed,

$$mv\times p = \text{constant},$$

where m is the mass of the planet and p is the perpendicular from S to OB produced.

$$\therefore \frac{m . \delta s . p}{\delta t} = \text{constant}.$$

But the area δA of the triangle SBO $= \frac{1}{2}$ base × height $= \delta s \times p/2$.

$$\therefore m.2\frac{\delta A}{\delta t} = \text{constant}$$

$$\therefore \frac{\delta A}{\delta t} = \text{constant},$$

since $2m$ is constant. Thus if the conservation of angular momentum is true, the area swept out per second by the radius SO is constant while the planet O moves in its orbit. In other words, equal areas are swept out in equal times. *But this is Kepler's second law*, which has been observed to be true for centuries (see p. 58). Consequently, the principle of the conservation of angular momentum has stood the test of time. From the equality of the angular momentum values at O and C, where p is less than p_1, it follows that v is greater than v_1. Thus the planet speeds up on approaching S.

The force on O is always one of attraction towards S. It is described as a *central force*. Thus the force has no moment about O and hence the angular momentum of the planet about S is conserved.

Kinetic Energy of a Rolling Object

When an object such as a cylinder or ball rolls on a plane, the object is rotating as well as moving bodily along the plane; therefore it has rotational energy as well as translational energy.

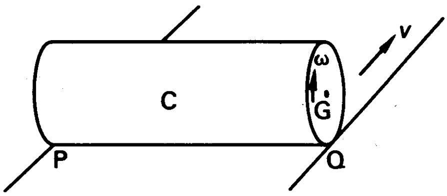

FIG. 3.13 Rolling object

Consider a cylinder C rolling along a plane without slipping, Fig. 3.13. At any instant the line of contact, PQ, with the plane is at rest, and we can consider the whole of the cylinder to be rotating about this axis. Hence the energy of the cylinder $= \frac{1}{2}I_1\omega^2$, where I_1 is the moment of inertia about PQ and ω is the angular velocity.

But if I is the moment of inertia about a parallel axis through the centre of gravity of the cylinder, M is the mass of the cylinder and a its radius, then

$$I_1 = I + Ma^2,$$

from the result on p. 79.

$$\therefore \text{energy of cylinder} = \tfrac{1}{2}(I + Ma^2)\omega^2$$

$$= \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}Ma^2\omega^2$$

$$\therefore \text{Energy} = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}Mv^2 \quad . \quad . \quad (1)$$

since, by considering the distance rolled and the angle then turned, $v = a\omega =$ velocity of centre of gravity. This energy formula is true for any moving object.

As an application of the energy formula, suppose a *ring* rolls along a plane. The moment of inertia about the centre of gravity, its centre, $= Ma^2$ (p. 76); also, the angular velocity, ω, about its centre $= v/a$, where v is the velocity of the centre of gravity.

$$\therefore \text{ kinetic energy of ring} = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}Ma^2 \times \left(\frac{v}{a}\right)^2 = Mv^2.$$

By similar reasoning, the kinetic energy of a sphere rolling down a plane

$$= \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}Mv^2 + \tfrac{1}{2} \times \tfrac{2}{5}Ma^2 \times \left(\frac{v}{a}\right)^2 = \tfrac{7}{10}Mv^2,$$

since $I = 2Ma^2/5$ (p. 78).

Acceleration of Rolling Object

We can now deduce the acceleration of a rolling object down an inclined plane.

As an illustration, suppose a solid cylinder rolls down a plane. Then

$$\text{kinetic energy} = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2.$$

But moment of inertia, I, about an axis through the centre of gravity parallel to the plane $= \tfrac{1}{2}Ma^2$, and $\omega = v/a$, where a is the radius.

$$\therefore \text{ kinetic energy} = \tfrac{1}{2}Mv^2 + \tfrac{1}{4}Mv^2 = \tfrac{3}{4}Mv^2.$$

If the cylinder rolls from *rest* through a distance s, the loss of potential energy $= Mgs \sin\alpha$, where α is the inclination of the plane to the horizontal.

$$\therefore \tfrac{3}{4}Mv^2 = Mgs \sin\alpha$$

$$\therefore v^2 = \frac{4g}{3}s \sin\alpha$$

But $\quad v^2 = 2as$, where a is the linear acceleration.

$$\therefore 2as = \frac{4g}{3}s \sin\alpha$$

$$\therefore a = \frac{2g}{3}\sin\alpha \qquad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

The acceleration if sliding, and no rolling, took place down the plane is $g \sin\alpha$. The cylinder has thus a smaller acceleration when rolling.

The time t taken to move through a distance s from rest is given by $s = \tfrac{1}{2}at^2$. Thus, from (i),

$$s = \tfrac{1}{3}gt^2 \sin\alpha,$$

$$\text{or} \quad t = \sqrt{\frac{3s}{g \sin\alpha}}.$$

If the cylinder is *hollow*, instead of solid as assumed, the moment of inertia about an axis through the centre of gravity parallel to the plane is greater than that for a solid cylinder, assuming the same mass and dimensions (p. 88). The time taken for a hollow cylinder to roll a given distance from rest on the plane is then greater than that taken by the solid cylinder, from reasoning similar to that above; and thus if no other means were available, a time test on an inclined plane will distinguish between a solid and a hollow cylinder of the same dimensions and mass. If a torsion wire is available, however, the cylinders can be suspended in turn, and the period of torsional oscillations determined. The cylinder of larger moment of inertia, the hollow cylinder, will have a greater period, as explained on p. 89.

Measurement of Moment of Inertia of Flywheel

The moment of inertia of a flywheel W about a horizontal axle A can be determined by tying one end of some string to a pin on the axle, winding the string round the axle, and attaching a mass M to the other end of the string, Fig. 3.14. The length of string is such that M reaches the floor, when released, at the same instant as the string is completely unwound from the axle.

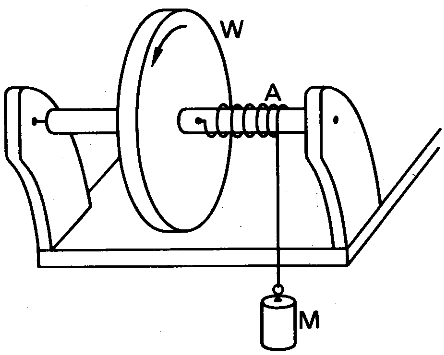

FIG. 3.14 Moment of inertia of flywheel

M is released, and the number of revolutions, n, made by the wheel W up to the occasion when M strikes the ground is noted. The further number of revolutions n_1 made by W until it comes finally to rest, and the time t taken, are also observed by means of a chalk-mark on W.

Now the loss in potential energy of M = gain in kinetic energy of M + gain in kinetic energy of flywheel + work done against friction.

$$\therefore\ Mgh = \tfrac{1}{2}Mr^2\omega^2 + \tfrac{1}{2}I\omega^2 + nf, \qquad \text{(i)}$$

where h is the distance M has fallen, r is the radius of the axle, ω is the angular velocity, I is the moment of inertia, and f is the energy per turn expended against friction. Since the energy of rotation of the flywheel when the mass M reaches the ground = work done against friction in n_1 revolutions, then

$$\tfrac{1}{2}I\omega^2 = n_1 f.$$

$$\therefore f = \tfrac{1}{2}\frac{I\omega^2}{n_1}.$$

Substituting for f in (i),

$$\therefore Mgh = \tfrac{1}{2}Mr^2\omega^2 + \tfrac{1}{2}I\omega^2\left(1+\frac{n}{n_1}\right) \quad . \quad . \quad \text{(ii)}$$

Since the angular velocity of the wheel when M reaches the ground is ω, and the final angular velocity of the wheel is zero after a time t, the average angular velocity $= \omega/2 = 2\pi n_1/t$. Thus $\omega = 4\pi n_1/t$. Knowing ω and the magnitude of the other quantities in (ii), the moment of inertia I of the fly-wheel can be calculated.

Period of Oscillation of Rigid Body

On p. 81 we showed that the moment of the forces acting on rotating objects $= Id\omega/dt = Id^2\theta/dt^2$, where I is the moment of inertia about the axis concerned and $d^2\theta/dt^2$ is the angular acceleration about the axis. Consider a rigid body oscillating about a fixed axis O, Fig. 3.15. The moment of the weight mg (the only external force) about O is $mgh \sin\theta$, or $mgh\theta$ if θ is small, where h is the distance of the centre of gravity from O.

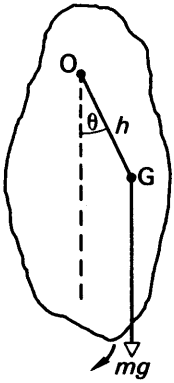

FIG. 3.15 Compound pendulum

$$\therefore I\frac{d^2\theta}{dt^2} = -mgh\theta,$$

the minus indicating that the moment due to the weight always *opposes* the growth of the angle θ.

$$\therefore \frac{d^2\theta}{dt^2} = \frac{-mgh}{I}\theta = -\omega^2\theta,$$

where $\omega^2 = mgh/I$.

$\therefore$ the motion is simple harmonic motion (p. 44),

and $$\text{period, } T, = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{mgh/I}} = 2\pi\sqrt{\frac{I}{mgh}} \quad . \quad . \quad (1)$$

If $I = mk_1^2$, where k_1 is the radius of gyration about O,

$$T = 2\pi\sqrt{\frac{mk_1^2}{mgh}} = 2\pi\sqrt{\frac{k_1^2}{gh}} \quad . \quad . \quad . \quad (2)$$

A *ring* of mass m and radius a will thus oscillate about an axis through a point O on its circumference normal to the plane of the ring with a period T given by

$$T = 2\pi\sqrt{\frac{I_0}{mga}}.$$

But $I_0 = I_G + ma^2$ (theorem of parallel axes) $= ma^2 + ma^2 = 2ma^2$.

$$\therefore T = 2\pi\sqrt{\frac{2ma^2}{mga}} = 2\pi\sqrt{\frac{2a}{g}}.$$

Thus if $a = 0{\cdot}5$ m, $g = 9{\cdot}8$ m s^{-2},

$$T = 2\pi\sqrt{\frac{2\times 0{\cdot}5}{9{\cdot}8}} = 2{\cdot}0 \text{ seconds (approx).}$$

Measurement of Moment of Inertia of Plate

The moment of inertia of a circular disc or other plate about an axis perpendicular to its plane, for example, can be measured by means of torsional oscillations. The plate is suspended horizontally from a vertical torsion wire, and the period T_1 of torsional oscillations is measured. Then, from (1),

$$T_1 = 2\pi\sqrt{\frac{I_1}{c}}, \qquad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where I_1 is the moment of inertia and c is the constant (opposing couple per unit radian) of the wire (p. 164). A ring or annulus of *known* moment of inertia I_2 is now placed on the plate concentric with the axis, and the new period T_2 is observed. Then

$$T_2 = 2\pi\sqrt{\frac{I_1+I_2}{c}} \qquad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

By squaring (i) and (ii), and then eliminating c, we obtain

$$I_1 = \frac{T_1^2}{T_2^2 - T_1^2}\cdot I_2.$$

Thus knowing T_1, T_2, and I_2, the moment of inertia I_1 can be calculated.

Compound Pendulum. Since $I = I_G + mh^2 = mk^2 + mh^2$, where I_G is the moment of inertia about the centre of gravity, h is the distance of the axis O from the centre of gravity, and k is the radius of gyration about the centre of gravity, then, from previous,

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{mk^2+mh^2}{mgh}}.$$

$$\therefore T = 2\pi\sqrt{\frac{k^2+h^2}{gh}}.$$

Hence $$T = 2\pi\sqrt{\frac{l}{g}},$$

where $$l = \frac{k^2+h^2}{h} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Thus $(k^2+h^2)/h$ is the length, l, of the *equivalent simple pendulum.*

From (i), $$h^2 - hl + k^2 = 0.$$

$$\therefore h_1 + h_2 = l, \text{ and } h_1h_2 = k^2,$$

where h_1 and h_2 are the roots of the equation.

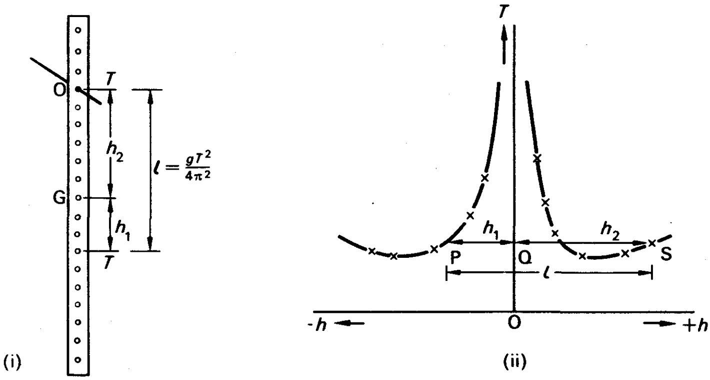

FIG. 3.16 Compound pendulum experiment

By timing the period of vibration, T, of a long rod about a series of axes at varying distances h on either side of the centre of gravity, and then plotting a graph of T v. h, two different values of h giving the same period can be obtained, Fig. 3.16 (i), (ii). Suppose h_1, h_2 are the two values. Then from the result just obtained, $h_1 + h_2 = l$, the length of the equivalent simple pendulum. Thus, since $T = 2\pi\sqrt{l/g}$,

$$g = \frac{4\pi^2 l}{T^2} = \frac{4\pi^2(h_1 + h_2)}{T^2}.$$

In Fig. 3.16 (ii), $PQ + QS = h_1 + h_2 = l$.

Kater's Pendulum. The acceleration due to gravity was first measured by the simple pendulum method, and calculated from the relation $g = 4\pi^2 l/T^2$, with the usual notation. The length l, the distance from the point of suspension to the centre of gravity of the bob, however, cannot be determined with very great accuracy.

In 1817 Captain Kater designed a reversible pendulum, with knife-edges for the suspension; it was a compound pendulum. Now it has just been shown that the same period is obtained between two non-symmetrical points on a compound pendulum when their distance apart is l, the length of the equivalent simple pendulum. Thus if T is the period about either knife-edge when this occurs, $g = 4\pi^2 l/T^2$, where l is now the distance between the knife-edges. The pendulum is made geometrically symmetrical about the mid-point, with a brass bob at one end and a wooden bob of the same size at the other. A movable large and small weight are placed between the knife-edges, which are about one metre apart. The period is then slightly greater than 2 seconds.

To find g, the pendulum is set up in front of an accurate seconds clock, with the bob of the clock and that of the Kater pendulum in line with each other, and both sighted through a telescope. The large weight on the pendulum is moved until the period is nearly the same about either knife-edge, and the small weight is used as a fine adjustment. When the periods are the same, the distance l between the knife-edges is measured very accurately by a comparator method with a microscope and standard metre. Thus knowing T and l, g can be calculated from $g = 4\pi^2 l/T^2$.

For details of the experiment the reader should consult *Advanced Practical Physics for Students* by Worsnop and Flint (Methuen).

Summary

The following table compares the translational (linear) motion of a small mass m with the rotational motion of a large object of moment of inertia I.

	Linear Motion	Rotational Motion
1.	Velocity, v	Velocity $v = r\omega$
2.	Momentum $= mv$	Angular momentum $= I\omega$
3.	Energy $= \frac{1}{2}mv^2$	Rotational energy $= \frac{1}{2}I\omega^2$
4.	Force, F, $= ma$	Torque, C, $= I \times$ ang. accn. $(d^2\theta/dt^2)$
5.	Simple pendulum: $T = 2\pi\sqrt{\frac{l}{g}}$	Compound pendulum: $T = 2\pi\sqrt{\frac{I}{mgh}}$
6.	Motion down inclined plane—energy equation: $\frac{1}{2}mv^2 = mgh \sin\theta$	Rotating without slipping down inclined plane—energy equation: $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh \sin\theta$
7.	Conservation of linear momentum on collision, if no external forces	Conservation of angular momentum on collision, if no external couple

EXAMPLE

What is meant by the *moment of inertia* of an object about an axis?

Describe and give the theory of an experiment to determine the moment of inertia of a flywheel mounted on a horizontal axle.

A uniform circular disc of mass 20 kg and radius 15 cm is mounted on a horizontal cylindrical axle of radius 1·5 cm and negligible mass. Neglecting frictional losses in the bearings, calculate (*a*) the angular velocity acquired from rest by the application for 12 seconds of a force of 2·0 kgf tangential to the axle, (*b*) the kinetic energy of the disc at the end of this period, (*c*) the time required to bring the disc to rest if a braking force of 0·1 kgf were applied tangentially to its rim. (*L.*)

Moment of inertia of disc, I, $= \frac{1}{2}Ma^2 = \frac{1}{2} \times 20\,(\text{kg}) \times 0{\cdot}15^2\,(\text{m}^2) = 0{\cdot}225\ \text{kg m}^2$.

(*a*) Torque due to 2 kgf tangential to axle

$$= 2 \times 9{\cdot}8\,(\text{N}) \times 0{\cdot}015\,(\text{m}) = 0{\cdot}294\ \text{N m}.$$

$$\therefore \text{angular acceleration} = \frac{\text{torque}}{I} = \frac{0{\cdot}294}{0{\cdot}225}\ \text{rad s}^{-2}.$$

$$\therefore \text{after 12 seconds, angular velocity} = \frac{12 \times 0{\cdot}294}{0{\cdot}225} = 15{\cdot}7\ \text{rad s}^{-1}.$$

(*b*) K.E. of disc after 12 seconds $= \frac{1}{2}I\omega^2$

$$= \frac{1}{2} \times 0{\cdot}225 \times 15{\cdot}7^2 = 27{\cdot}8\ \text{J}.$$

(*c*) Decelerating torque $= 0{\cdot}1 \times 9{\cdot}8\,(\text{N}) \times 0{\cdot}15\,(\text{m})$.

$$\therefore \text{angular deceleration} = \frac{\text{torque}}{I} = \frac{0{\cdot}1 \times 9{\cdot}8 \times 0{\cdot}15}{0{\cdot}225}\ \text{rad s}^{-2}.$$

$$\therefore \text{time to bring disc to rest} = \frac{\text{initial angular velocity}}{\text{angular deceleration}}$$

$$= \frac{15{\cdot}7 \times 0{\cdot}225}{0{\cdot}1 \times 9{\cdot}8 \times 0{\cdot}15} = 24\ \text{seconds}.$$

EXERCISES 3

(Assume $g = 10\ m\ s^{-2}$ unless otherwise stated)

What are the missing words in the statements 1–4?

1. The kinetic energy of an object rotating about an axis is calculated from . . .

2. The angular momentum of the object is calculated from . . .

3. '$I \times$ angular acceleration' is equal to the . . . on the object.

4. The period of oscillation of an object about an axis is calculated from . . .

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 5–8?

5. When a sphere of moment of inertia I about its centre of gravity, and mass m, rolls from rest down an inclined plane without slipping, its kinetic energy is calculated from $A\ \frac{1}{2}I\omega^2$, $B\ \frac{1}{2}mv^2$, $C\ I\omega+mv$, $D\ \frac{1}{2}I\omega^2+\frac{1}{2}mv^2$, $E\ I\omega$.

6. If a hoop of radius a oscillates about an axis through its circumference perpendicular to its plane, the period is $A\ 2\pi\sqrt{a/g}$, $B\ 2\pi\sqrt{2a/g}$, $C\ 2\pi\sqrt{g/a}$, $D\ 2\pi\sqrt{g/2a}$, $E\ a/2$.

7. Planets moving in orbit round the sun *A* increase in velocity at points near the sun because their angular momentum is constant, *B* increase in velocity near the sun because their energy is constant, *C* decrease in velocity near the sun owing to the increased attraction, *D* sweep out equal mass in equal times because their energy is constant, *E* always have circular orbits.

8. If a constant couple of 500 newton metre turns a wheel of moment of inertia 100 kg m^2 about an axis through its centre, the angular velocity gained in two seconds is *A* 5 rad s^{-1}, *B* 100 m s^{-1}, *C* 200 m s^{-1}, *D* 2 m s^{-1}, *E* 10 rad s^{-1}.

9. A uniform rod has a mass of 60 g and a length 20 cm. Calculate the moment of inertia about an axis perpendicular to its length (i) through its centre, (ii) through one end. Prove the formulae used.

10. What is the *Theorem of Parallel Axes*? A uniform disc has a mass of 4 kg and a radius of 2 m. Calculate the moment of inertia about an axis perpendicular to its plane (i) through its centre, (ii) through a point of its circumference.

11. What is the *Theorem of Perpendicular Axes*? A ring has a radius of 20 cm and a mass of 100 g. Calculate the moment of inertia about an axis (i) perpendicular to its plane through its centre, (ii) perpendicular to its plane passing through a point on its circumference, (iii) in its plane passing through the centre.

12. What is the formula for the kinetic energy of (i) a particle, (ii) a rigid body rotating about an axis through its centre of gravity, (iii) a rigid body rotating about an axis through any point? Calculate the kinetic energy of a disc of mass 5 kg and radius 1 m rolling along a plane with a uniform velocity of 2 m s^{-1}.

13. A sphere rolls down a plane inclined at 30° to the horizontal. Find the acceleration and velocity of the sphere after it has moved 5·0 m from rest along the plane, assuming the moment of inertia of a sphere about a diameter is $2Ma^2/5$, where M is the mass and a is the radius.

14. A uniform rod of length 3·0 m is suspended at one end so that it can move about an axis perpendicular to its length, and is held inclined at 60° to the vertical and then released. Calculate the angular velocity of the rod when (i) it is inclined at 30° to the vertical, (ii) reaches the vertical.

15. Define the *moment of inertia* of a rigid object about an axis.

A ring of radius 2·0 m oscillates about an axis on its circumference which is perpendicular to the plane of the ring. Calculate the period of oscillation. Give an explanation of any formula used.

16. A flywheel with an axle 1·0 cm in diameter is mounted in frictionless bearings and set in motion by applying a steady tension of 200 gf to a thin thread wound tightly round the axle. The moment of inertia of the system about its axis of rotation is $5{\cdot}0 \times 10^{-4}$ kg m^2. Calculate (*a*) the angular acceleration of the flywheel when 1 m of thread has been pulled off the axle, (*b*) the constant retarding couple which must then be applied to bring the flywheel to rest in one complete turn, the tension in the thread having been completely removed. (*N.*)

17. Define the moment of inertia of a body about a given axis. Describe how the moment of inertia of a flywheel can be determined experimentally.

A horizontal disc rotating freely about a vertical axis makes 100 r.p.m. A small piece of wax of mass 10 g falls vertically on to the disc and adheres to it at a distance of 9 cm from the axis. If the number of revolutions per minute is thereby reduced to 90, calculate the moment of inertia of the disc. (*N.*)

18. Describe an experiment using a bar pendulum to determine the acceleration due to gravity. Show how the result is calculated from the observations.

A uniform disc of diameter 12·0 cm and mass 810 g is suspended with its plane horizontal by a torsion wire and allowed to perform small torsional oscillations about a vertical axis through its centre. The disc is then replaced by a uniform sphere which is allowed to oscillate similarly about a diameter. If the period of oscillation of the sphere is 1·66 times that of the disc, determine the moment of inertia of the sphere about a diameter. (*L.*)

19. A plane sheet of metal of uniform thickness and of irregular shape is pierced with a number of small holes, irregularly distributed, so that it can be pivoted about an axis through any one of them to swing in its own plane. Describe how you would proceed in order to find the value of the moment of inertia of the sheet about an axis through its centre of gravity normal to its plane.

A rigid bar pendulum is pivoted at a distance h from its centre of gravity. When a piece of lead of negligible size and of mass M equal to that of the pendulum, is attached at the centre of gravity the periodic time of the pendulum is reduced to 0·80 of its former value. Find an expression for the moment of inertia of the pendulum about the pivot. (*L.*)

20. Define *moment of inertia* and derive an expression for the kinetic energy of a rigid body of moment of inertia I about a given axis when it is rotating about that axis with a uniform angular velocity ω.

Give two examples of physical phenomena in which moment of inertia is a necessary concept for a theoretical description, in each case showing how the concept is applied.

A uniform spherical ball starts from rest and rolls freely without slipping down an inclined plane at 10° to the horizontal along a line of greatest slope. Calculate its velocity after it has travelled 5 m. (M.I. of sphere about a diameter = $2Mr^2/5$.) (*O. & C.*)

21. Explain the meaning of the term *moment of inertia*. Describe in detail how you would find experimentally the moment of inertia of a bicycle wheel about the central line of its hub.

A uniform cylinder 20 cm long, suspended by a steel wire attached to its mid-point so that its long axis is horizontal, is found to oscillate with a period of

2 seconds when the wire is twisted and released. When a small thin disc, of mass 10 g, is attached to each end the period is found to be 2·3 seconds. Calculate the moment of inertia of the cylinder about the axis of oscillation. (*N.*)

22. What is meant by 'moment of inertia'? Explain the importance of this concept in dealing with problems concerning rotating bodies.

Describe, with practical details, how you would determine whether a given cylindrical body were hollow or not without damaging it. (*C.*)

23. Define *moment of inertia*, and find an expression for the kinetic energy of a rigid body rotating about a fixed axis.

A sphere, starting from rest, rolls (without slipping) down a rough plane inclined to the horizontal at an angle of 30°, and it is found to travel a distance of 13·5 m in the first 3 seconds of its motion. Assuming that F, the frictional resistance to the motion, is independent of the speed, calculate the ratio of F to the *weight* of the sphere. (For a sphere of mass m and radius r, the moment of inertia about a diameter is $\frac{2}{5}mr^2$.) (*O. & C.*)