chapter twenty-seven

Vibrations in Pipes, Strings, Rods

Introduction

THE music from an organ, a violin, or a xylophone is due to vibrations in the air set up by oscillations in these instruments. In the organ, air is blown into a pipe, which sounds its characteristic note as the air inside it vibrates; in the violin, the strings are bowed so that they oscillate; and in a xylophone a row of metallic rods are struck in the middle with a hammer, which sets them into vibration.

Before considering each of the above cases in more detail, it would be best to consider the feature common to all of them. A violin string is fixed at both ends, A, B, and waves travel along *m*, *n* to each end of the string when it is bowed and are there reflected, Fig. 27.1 (i).

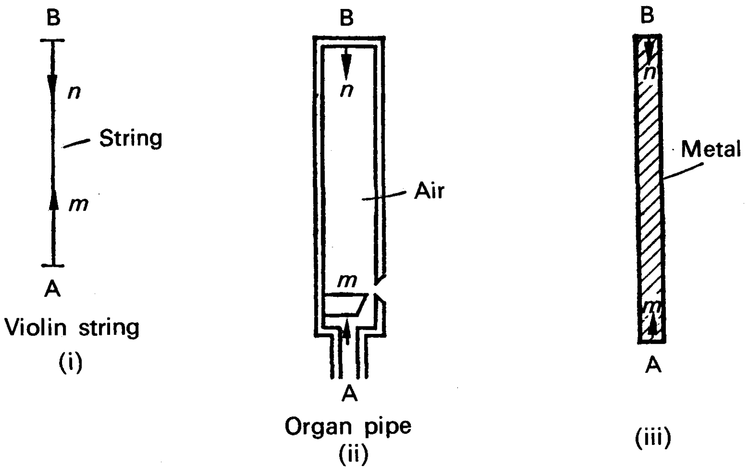


FIG. 27.1. Reflection of waves in instruments.

The vibrations of the particles of the string are hence due to *two waves of the same frequency and amplitude travelling in opposite directions.* A similar effect is obtained with an organ pipe closed at one end B, Fig. 27.1 (ii). If air is blown into the pipe at A, a wave travels along the direction *m* and is reflected at B in the opposite direction *n*. The vibrations of the air in the pipe are thus due to two waves travelling in opposite directions. If a metal rod is fixed at its middle in a vice and stroked at one end A, a wave travels along the rod in the direction *m* and is reflected at the other end B in the direction *n*, Fig. 27.1 (iii). The vibrations of the rod, which produce a high-pitched note, are thus due to two waves travelling in opposite directions.

The resultant effect of two waves travelling in opposite directions with equal amplitude and frequency can easily be demonstrated. A light string, or thread, is tied to the end of a clapper, P, of an electric bell, and the other end of the string is passed round a grooved wheel, Fig. 27.2.

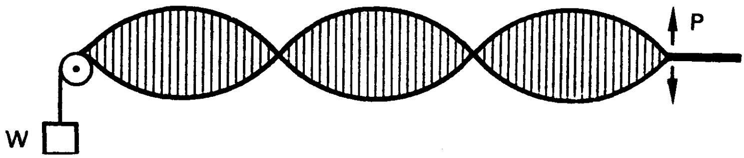


FIG. 27.2. Demonstration of stationary wave.

When the clapper vibrates, and a suitable weight W is attached to the string, a number of *stationary loops* is observed along the vibrating string, somewhat as shown in Fig. 27.2. By altering W a different number of stationary loops can be obtained. The wave along the string is known as a *stationary wave,* and we shall now discuss the formation of a stationary wave in detail.

Stationary Waves

Consider a plane-progressive wave *a* travelling in air along OA, Fig. 27.3. If it meets a wall at W a reflected wave *b* is obtained, and the condition of the air along W is due to the combined effects of *a, b.*

The layer of air at W must always be at rest since it is in contact with a fixed wall. For convenience, suppose that the displacements of the layers of air due to *a* at the instant shown are those represented by the sine wave in Fig. 27.3 (i), so that the displacement of the layer at W due to the incident wave is a maximum. Since the layer at W is always at rest, the displacements of the layers due to the wave *reflected* from the wall must be represented by the curve *b* at the same instant; otherwise the net displacement at W, which is the algebraic sum of WR, WH, will not be zero. From the curves *a, b* shown in Fig. 27.3 (i), it follows that the wave *b* reflected by the wall is 180° out of phase with the incident wave *a.*

At the instant *t* represented in Fig. 27.3 (i), the algebraic sum, S, of the displacements of the layers *everywhere* along OW is zero if the amplitudes of *a, b* are equal and the curves have the same wavelength. At an instant *T*/4 later, where *T* is the period of vibration of the layers the displacements of the layer due to the incident and reflected waves are those shown in Fig. 27.3 (ii). This can best be understood by imagining the incident wave *a* to have advanced to the right by ¼-wavelength, and the reflected wave to have advanced to the left by ¼-wavelength, which implies that the vibrating layers have now reached a displacement corresponding to a time *T*/4 later than *t*. The algebraic sums S of the displacement is then represented by the curve S in Fig. 27.3 (ii). At the end of a further time *T*/4, the displacements due to *a, b* are those shown

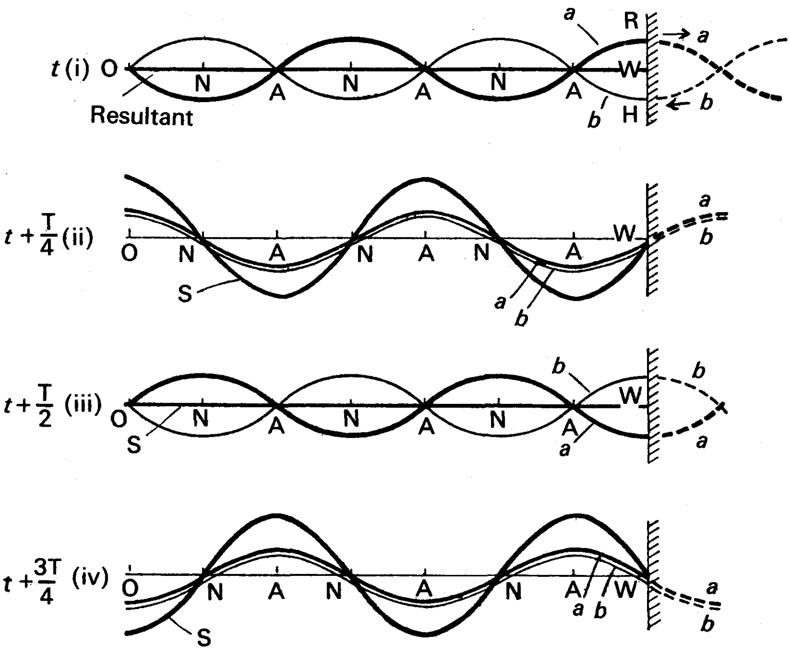


FIG. 27.3. Formation of stationary waves.

in Fig. 27.3 (iii); the waves have now advanced another ¼-wavelength in opposite directions. The algebraic sum S of the displacements is again zero everywhere along OW at this instant. After a further time $T/4$, the displacements of the layers, and the resultant displacement S, are those shown in Fig. 27.3 (iv). The wave in the air represented by S is called a stationary wave.

Nodes and Antinodes

We have now sufficient information to deduce the conditions of the layers of air along OW when a stationary wave is obtained. From the curves showing the resultant displacement, S, in Fig. 27.3, it can be seen that some layers, marked N, are *permanently* at rest; these are known as **nodes.** The layers marked A, however, are vibrating through an amplitude twice as big as the incident or reflected waves, see Fig. 27.3 (ii) and (iv), and these are known as **antinodes.** Layers between consecutive nodes are vibrating in phase with each other, but the amplitude of vibration varies from zero at a node N to a maximum

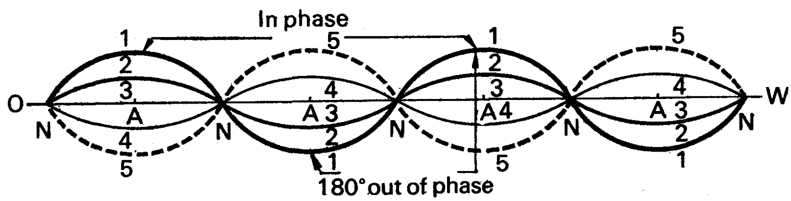


FIG. 27.4. Nodes and antinodes.

at the antinode A. Fig. 27.4 represents the displacement of the layers along OW at five different instants 1, 2, 3, 4, 5. It follows that

the distance between consecutive nodes, $NN = \frac{\lambda}{2}$ . . (i)

where λ is the wavelength of the stationary wave;

the distance between consecutive antinodes, $AA = \frac{\lambda}{2}$, . (ii)

and

the distance from a node to the next antinode, $NA = \frac{\lambda}{4}$ . . (iii)

The importance of the nodes and antinodes in a stationary wave lies in their simple connection with the wavelength.

Differences Between Plane-Progressive and Stationary Waves

At the beginning of Chapter 25, we considered in detail the plane-progressive wave and its effect on the medium (pp. 584 and 587). It was then shown that each layer vibrates with constant amplitude at the same frequency, and that each layer is out of phase with others near to it. When a stationary wave is present in a medium, however, some layers (nodes) are permanently at rest; others between the nodes are vibrating in phase with different amplitudes, increasing to a maximum at the antinodes. A stationary wave is always set up when two plane-progressive waves of equal amplitude and frequency travel in opposite directions in the same medium.

Mathematical proof of stationary wave properties. The properties of the stationary wave, already deduced, can be obtained easily from a mathematical treatment. Suppose $y_1 = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$ is a plane-progressive wave travelling in one direction along the x-axis (p. 587). Then $y_2 = a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)$ represents a wave of the same amplitude and frequency travelling in the opposite direction. The resultant displacement, y, is hence given by

$$y = y_1 + y_2 = a \left[\sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) + \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda}\right)\right]$$

from which $$y = 2a \sin \frac{2\pi t}{T} \, . \cos \frac{2\pi x}{\lambda} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

using the transformation of the sum of two sine functions to a product.

$$\therefore \quad y = B \sin \frac{2\pi t}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where $$B = 2a \cos \frac{2\pi x}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

From (ii), B is the magnitude of the *amplitude* of vibration of the various layers; and from (iii) it also follows that the amplitude is a maximum and equal to $2a$ at $x = 0$, $x = \lambda/2$, $x = \lambda$, and so on. These points are thus antinodes, and consecutive antinodes are hence separated by a distance $\lambda/2$. The amplitude B is zero when $x = \lambda/4$, $x = 3\lambda/4$, $x = 5\lambda/4$, and so on. These points are thus nodes, and they are hence midway between consecutive antinodes.

The particle velocity in a stationary wave is the rate of change of the displacement (y) of the particle with respect to time (t). The velocity at the nodes is always zero since the particles there are permanently at rest. The velocity at an antinode increases from zero (when the particle is at the end of its oscillation) to a maximum (when the particle passes through its mean or original position). The corresponding displacement and velocity curves in the latter case are illustrated in Fig. 27.5 by P, M

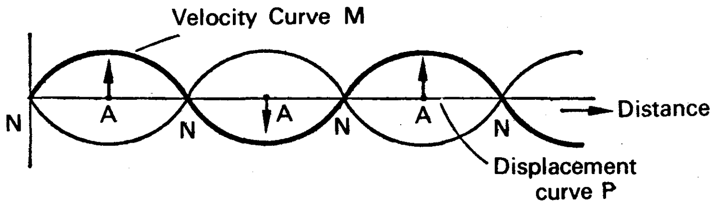


FIG. 27.5. Particle velocity due to stationary wave.

respectively. It will be noted that particles at neighbouring antinodes are moving in opposite directions at any instant.

Variation of Pressure in the Stationary Wave

Having considered the variation of the displacements and the particle velocities when a stationary wave travels in air, we must now turn our attention to the variation of *pressure* in the air.

Suppose that curve 1 represents the displacements at the antinodes and other points at an instant when they are a maximum, Fig. 27.6 (i). The layer of air immediately to the left of the node at a is then displaced towards a, since the displacement is positive from curve 1, and the layer immediately to the right of a also displaced towards a. The air at a is thus compressed, and the pressure is thus greater than normal, as represented by curve 1 in Fig. 27.6 (ii). The displacements of the layers on

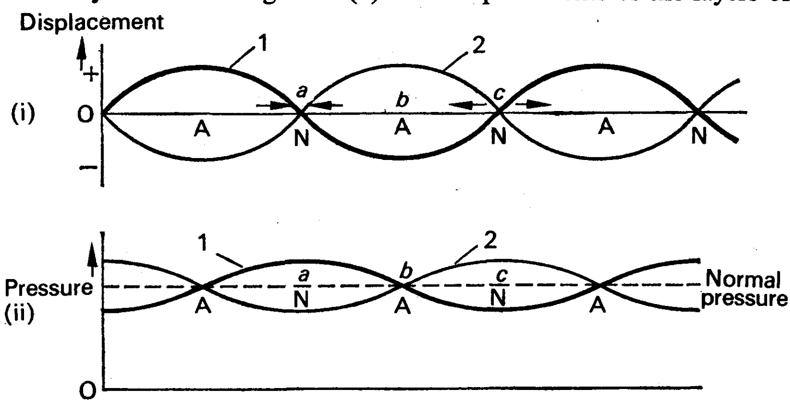


FIG. 27.6. Pressure variation due to stationary wave.

either side of the antinode at b are each a maximum to the left, and hence the pressure of the air is normal. The air on the left of the node c is displaced away from *a*, and the air on the right of *c* is also displaced away from *c*. The air is thus rarified here, and hence the pressure is less normal. By carrying out the same procedure at other points in the air, it can be seen that the pressure variation corresponds to the curve 1 in Fig. 27.6 (ii).

When the displacements change to those represented by curve 2 in Fig. 27.6 (i), the variation of pressure at the same instant is shown by curve 2 in Fig. 27.6 (ii). We can now see that *the pressure variation is always a maximum at a node of the stationary wave, and is always zero at an antinode of the stationary wave.* In a plane-progressive wave, however, the pressure variation is the same at every point in a medium (p. 585).

EXAMPLE

Distinguish between progressive and stationary wave motion. Describe and illustrate with an example how stationary wave motion is produced. Plane sound waves of frequency 100 Hz fall normally on a smooth wall. At what distances from the wall will the air particles have (*a*) maximum, (*b*) minimum amplitude of vibration? Give reasons for your answer. (The velocity of sound in air may be taken as 340 m s^{-1}.) (*L.*)

First part. See p. 587 and p. 643.

Second part. A stationary wave is set up between the source and the wall, due to the production of a reflected wave. The wall is a node, since the air in contact with it cannot move; and other nodes are at equal distances, *d*, from the wall. But if the wavelength is λ,

$$d = \frac{\lambda}{2} \text{ (p. 643).}$$

Also

$$\lambda = \frac{V}{f} = \frac{340}{100} = 3{\cdot}4 \text{ m}$$

$$\therefore \quad d = \frac{3{\cdot}4}{2} = 1{\cdot}7 \text{ m.}$$

Thus minimum amplitude of vibration is obtained 1·7, 3·4, 5·1 m . . . from the wall.

The antinodes are midway between the nodes. Thus maximum amplitude of vibration is obtained 0·85, 2·55, 4·25m, . . . from the wall.

VIBRATIONS OF AIR IN PIPES

Closed Pipe

A *closed* or *stopped organ pipe* consists essentially of a metal pipe closed at one end Q, and a blast of air is blown into it at the other end P, Fig. 27.7 (i). A wave thus travels up the pipe to Q, and is reflected at this end down the pipe, so that a *stationary wave* is obtained. The end Q of the closed pipe must be a node N, since the layer in contact with Q must be permanently at rest, and the open end A, where the air is free to vibrate, must be an antinode A. The simplest stationary wave in the air in the pipe is hence represented by g in Fig. 27.7 (ii), where the pipe is positioned horizontally to show the relative displacement, y, of the layers at different distances, x, from the closed end Q; the axis of the stationary wave is Qx.

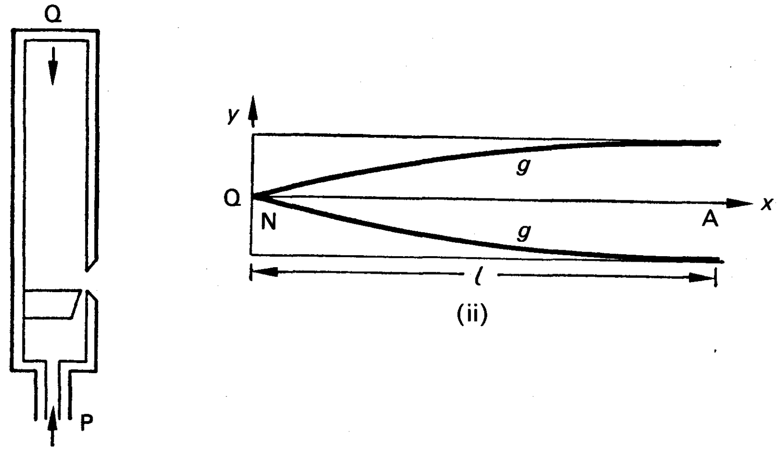


FIG. 27.7. (i). Closed (stopped) pipe. (ii). Fundamental of closed (stopped) pipe.

It can now be seen that the length l of the pipe is equal to the distance between a node N and a consecutive antinode A of the stationary wave. But NA $= \lambda/4$, where λ is the wavelength (p. 643).

$$\therefore \quad \frac{\lambda}{4} = l$$

$$\therefore \quad \lambda = 4l$$

But the frequency, f, of the note is given by $f = V/\lambda$, where V is the velocity of sound in air.

$$\therefore \quad f = \frac{V}{4l}$$

This is the frequency of the lowest note obtainable from the pipe, and it is known as its *fundamental.* We shall denote the fundamental frequency by f_0, so that

$$f_0 = \frac{V}{4l} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Overtones of Closed Pipe

If a stronger blast of air is blown into the pipes, notes of higher frequency can be obtained which are simple multiples of the fundamental frequency f_0. Two possible cases of stationary waves are shown in Fig. 27.8. In each, the closed end of the pipe is a node, and the open end is an antinode. In Fig. 27.8 (i), however, the length l of the pipe is

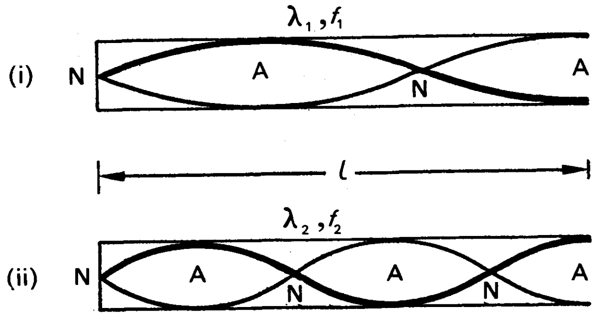


FIG. 27.8. Overtones in closed pipe.

related to the wavelength λ_1 of the wave by

$$l = \tfrac{3}{4}\lambda_1$$

$$\therefore \quad \lambda_1 = \frac{4l}{3}$$

The frequency f_1 of the note is thus given by

$$f_1 = \frac{V}{\lambda_1} = \frac{3V}{4l} \quad . \quad . \quad \text{(i)}$$

But

$$f_0 = \frac{V}{4l}$$

$$\therefore \quad f_1 = 3f_0 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

In Fig. 27.8 (ii), when a note of frequency f_2 is obtained, the length l of the pipe is related to the wavelength λ_2 by

$$l = \frac{5\lambda_2}{4}$$

$$\therefore \quad \lambda_2 = \frac{4l}{5}$$

$$\therefore \quad f_2 = \frac{V}{\lambda_2} = \frac{5V}{4l} \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$\therefore \quad f_2 = 5f_0 \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

By drawing other sketches of stationary waves, with the closed end as a node and the open end as an antinode, it can be shown that higher frequencies can be obtained which have frequencies of $7f_0$, $9f_0$, and so on. They are produced by blowing harder at the open end of the pipe. The frequencies obtainable at a closed pipe are hence f_0, $3f_0$, $5f_0$, and so on, i.e., the closed pipe gives only odd harmonics, and hence the frequencies $3f_0$, $5f_0$, etc. are possible *overtones*.

Open Pipe

An "open" pipe is one which is open at both ends. When air is blown into it at P, a wave m travels to the open end Q, where it is reflected in the direction n on encountering the free air, Fig. 27.9 (i). A stationary wave is hence set up in the air in the pipe, and as the two ends of the pipe are open, they must both be *antinodes*. The simplest type of wave is hence that shown in Fig. 27.9 (ii), the x-axis of the wave being drawn along the middle of the pipe, which is horizontal. A node N is midway between the two antinodes.

The length l of the pipe is the distance between consecutive antinodes. But the distance between consecutive antinodes $= \lambda/2$, where λ is the wavelength (p. 643).

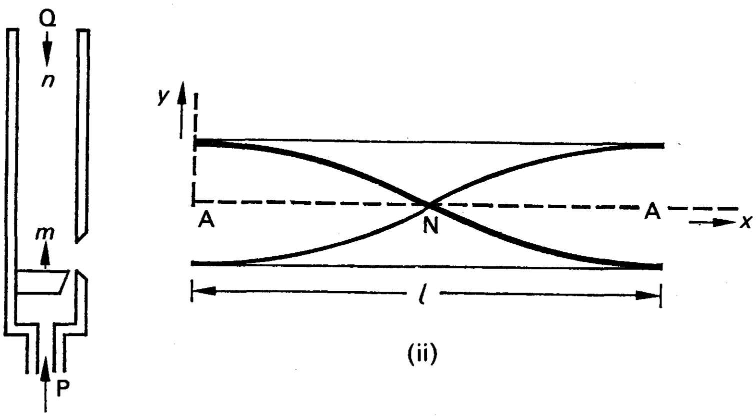


FIG. 27.9. (i). Open pipe. (ii). Fundamental of open pipe.

$$\therefore \quad \frac{\lambda}{2} = l$$

$$\therefore \quad \lambda = 2l$$

Thus the frequency f_0 of the note obtained from the pipe is given by

$$f_0 = \frac{V}{\lambda} = \frac{V}{2l} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the frequency of the fundamental note of the pipe.

Overtones of Open Pipe

Notes of higher frequencies than f_0 can be obtained from the pipe by blowing harder. The stationary wave in the pipe has always an antinode A at each end, and Fig. 27.10 (i) represents the case of a note of a frequency f_1.

The length l of the pipe is equal to the wavelength λ_1 of the wave in this case. Thus

$$f_1 = \frac{V}{\lambda_1} = \frac{V}{l}$$

But $$f_0 = \frac{V}{2l}, \text{ from (2) above.}$$

$$\therefore \quad f_1 = 2f_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

In Fig. 27.10 (ii), the length $l = \frac{3}{2}\lambda_2$, where λ_2 is the wavelength in the pipe. The frequency f_2 is thus given by

$$f_2 = \frac{V}{\lambda_2} = \frac{3V}{2l}$$

as $\lambda_2 = \frac{2l}{3}$.

$$\therefore \quad f_2 = 3f_0 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

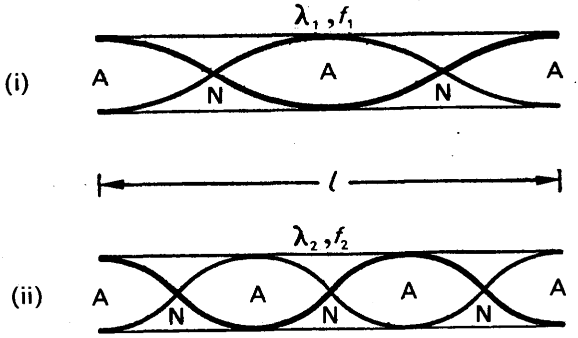


FIG. 27.10. Overtones of open pipes.

The frequencies of the overtones in the open pipe are thus $2f_0$, $3f_0$, $4f_0$, and so on, i.e., all harmonics are obtainable. The frequencies of the overtones in the closed pipe are $3f_0$, $5f_0$, $7f_0$, and so on, and hence the *quality* of the same note obtained from a closed and an open pipe is different (see p. 611).

Detection of Nodes and Antinodes, and Pressure Variation, in Pipes

The *nodes and antinodes* in a sounding pipe have been detected by suspending inside it a very thin piece of paper with lycopodium or fine sand particles on it, Fig. 27.11 (i). The particles are considerably agitated at the antinodes, but they are motionless at the nodes.

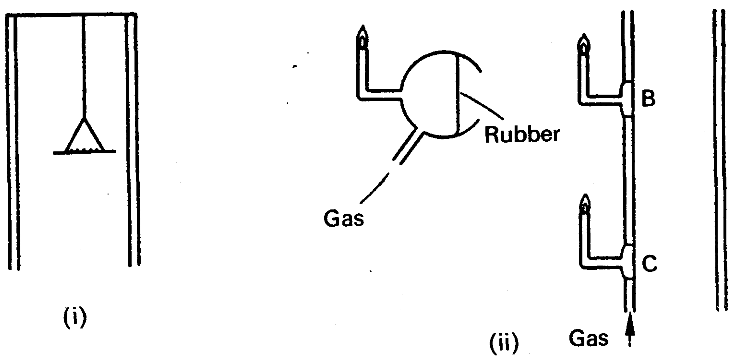


FIG. 27.11. (i). Detection of nodes and antinodes.
(ii). Detection of pressure.

The *pressure variation* in a sounding pipe has been examined by means of a sensitive flame, designed by Lord Rayleigh. The length of the flame can be made sensitive to the pressure of the gas supplied, so that if the pressure changes the length of flame is considerably affected. Several of the flames can be arranged at different parts of the pipe, with a thin rubber or mica diaphragm in the pipe, such as at B, C, Fig. 27.11 (ii). At a place of maximum pressure variation, which is a node (p. 645), the length of flame alters accordingly. At a place of constant (normal) pressure, which is an antinode, the length of flame remains constant.

The pressure variation at different parts of a sounding pipe can also be examined by using a suitable small microphone at B, C, instead of a flame. The microphone is coupled to a cathode-ray tube and a wave of maximum amplitude is shown on the screen when the pressure variation is a maximum. At a place of constant (normal) pressure, no wave is observed on the screen.

End-correction of Pipes

The air at the open end of a pipe is free to move, and hence the vibrations at this end of a sounding pipe extend a little into the air outside the pipe. The antinode of the stationary wave due to any note is thus a distance c from the open end in practice, known as the *end-correction*, and hence the wavelength λ in the case of a closed pipe is given by $\lambda/4 = l + c$, where l is the length of the pipe, Fig. 27.12 (i). In the case of an open pipe sounding its fundamental note, the wavelength λ is given by $\lambda/2 = l + c + c$, since *two* end-corrections are required, assuming the end-corrections are equal, Fig. 27.12 (ii). Thus $\lambda = 2(l + 2c)$. See also p. 648.

The mathematical theory of the end-correction was developed independently by Helmholtz and Rayleigh. It is now generally accepted

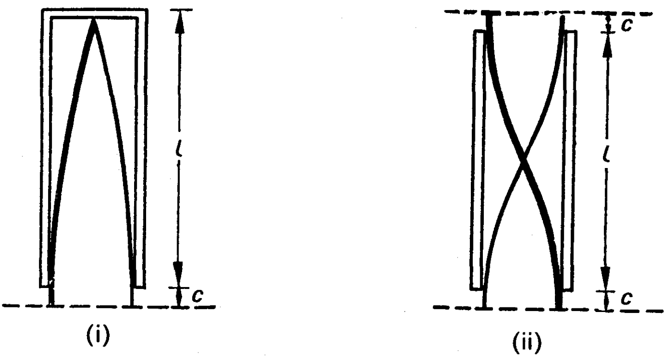


FIG. 27.12. (i). Closed pipe. (ii). Open pipe.

that $c = 0{\cdot}58r$, or $0{\cdot}6r$, where r is the radius of the pipe, so that the wider the pipe, the greater is the end-correction. It was also shown that the end-correction depends on the wavelength λ of the note, and tends to vanish for very short wavelengths.

Effect of Temperature, and End-correction, on the Pitch of Pipes

The frequency, f_0, of the fundamental note of a closed pipe of length l and end-correction c is given by

$$f_0 = \frac{V}{\lambda} = \frac{V}{4(l+c)} \quad . \quad . \quad . \quad \text{(i)}$$

with the usual notation, since $\lambda = 4(l + c)$. See p. 650. Now the velocity of sound, V, in air at $t°$ C is related to its velocity V_0 at 0° C by

$$\frac{V}{V_0} = \sqrt{\frac{273 + t}{273}} = \sqrt{1 + \frac{t}{273}} \quad . \quad . \quad \text{(ii)}$$

since the velocity is proportional to the square root of the absolute temperature. Substituting for V in (i),

$$\therefore \; f_0 = \frac{V_0}{4(l+c)} \sqrt{1 + \frac{t}{273}} \quad . \quad . \quad . \quad \text{(iii)}$$

From (iii), it follows that, with a given pipe, *the frequency of the fundamental increases as the temperature increases.* Also, for a given temperature and length of pipe, the frequency decreases as c increases. Now $c = 0{\cdot}6r$, where r is the radius of the pipe. Thus *the frequency of the note from a pipe of given length is lower the wider the pipe*, the temperature being constant. The same results hold for an open pipe.

Resonance

If a diving springboard is bent and then allowed to vibrate freely, it oscillates with a frequency which is called its *natural frequency*. When a diver on the edge of the board begins to jump up and down repeatedly, the board is forced to vibrate at the frequency of the jumps; and at first, when the amplitude is small, the board is said to be undergoing *forced vibrations*. As the diver jumps up and down to gain increasing height for his dive, the frequency of the periodic downward force reaches a stage where it is practically the same as the natural frequency of the board. The amplitude of the board then becomes very large, and the periodic force is said to have set the board in *resonance*.

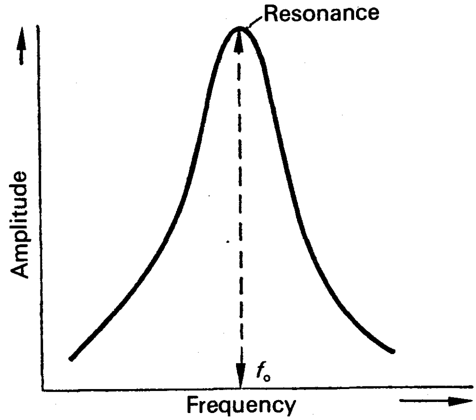


FIG. 27.13. Resonance curve.

A mechanical system which is free to move, like a wooden bridge or the air in pipes, has a natural frequency of vibration, f_0, which depends on its dimensions. When a periodic force of a frequency different from f_0 is applied to the system, the latter vibrates with a small amplitude and undergoes forced vibrations. When the periodic force has a frequency equal to the natural frequency f_0 of the system, the amplitude of vibration becomes a maximum, and the system is then set into resonance. Fig. 27.13 is a typical curve showing the variation of amplitude with frequency. Some time ago it was reported in the newspapers that a soprano who was broadcasting had broken a glass tumbler on the table of a listener when she had reached a high note. This is an example of resonance. The glass had a natural frequency equal to that of the note sung, and was thus set into a vibration sufficiently violent to break it.

The phenomenon of resonance occurs in other branches of Physics than Sound and Mechanics. When an electrical circuit containing a coil and capacitor is "tuned" to receive the radio waves from a distant transmitter, the frequency of the radio waves is equal to the natural frequency of the circuit and resonance is therefore obtained. A large current then flows in the electrical circuit. A dark line in a continuous spectrum, an absorption line, is an example of *optical resonance.* Thus some of the yellow wavelengths from the sun's spectrum are absorbed by molecules of sodium vapour in the cooler part of the sun's atmosphere, which are set into resonance (see p. 464).

Sharpness of resonance. As the resonance condition is approached, the effect of the damping forces on the amplitude increases. Damping prevents the amplitude from becoming infinitely large at resonance. The lighter the damping, the sharper is the resonance, that is, the amplitude diminishes considerably at a frequency slightly different from the resonant frequency, Fig. 27.14. A heavily-damped system has a fairly flat resonance curve. Tuning is therefore more difficult in a system which has light damping.

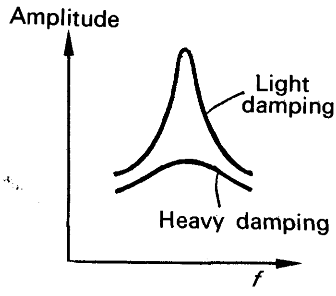


FIG. 27.14. Sharpness of resonance.

The effect of damping can be illustrated by attaching a simple pendulum carrying a pith bob, and one of the same length carrying a lead bob of equal size, to a horizontal string. The pendula are set into vibration by a third pendulum of equal length attached to the same string, and it is then seen that the amplitude of the lead bob is much greater than that of the pith bob. The damping of the pith bob due to air resistance is much greater than for the lead bob.

Resonance in a Tube or Pipe

If a person blows gently down a pipe closed at one end, the air inside vibrates freely, and a note is obtained from the pipe which is its fundamental (p. 646). A stationary wave then exists in the pipe, with a node N at the closed end and an antinode A at the open end, as explained on p. 645.

FIG. 27.15. Resonance in closed pipe.

If the prongs of a tuning-fork are held over the top of the pipe, the air inside it is set into vibration by the periodic force exerted on it by the prongs. In general, however, the vibrations are feeble, as they are *forced* vibrations, and the intensity of the sound heard is correspondingly small. But when a tuning-fork of the same frequency as the fundamental frequency of the pipe is held over the latter, the air inside is set in *resonance* by periodic force, and the amplitude of the vibrations is large. A loud note, which has the same frequency as the fork, is then heard coming from the pipe, and a stationary wave is set up with the top of the pipe acting as an antinode and the fixed end as a node, Fig. 27.15. If a sounding tuning-fork is held over a pipe open at both ends, resonance occurs when the stationary wave in the pipe has antinodes at the two open ends, as shown by Fig. 27.9; the frequency of the fork is then equal to the frequency of the fundamental of the open pipe.

Resonance Tube Experiment. Measurement of Velocity of Sound and "End-Correction" of Tube

If a sounding tuning-fork is held over the open end of a tube T filled with water, resonance is obtained at some position as the level of water is gradually lowered, Fig. 27.16 (i). The stationary wave set up is then as

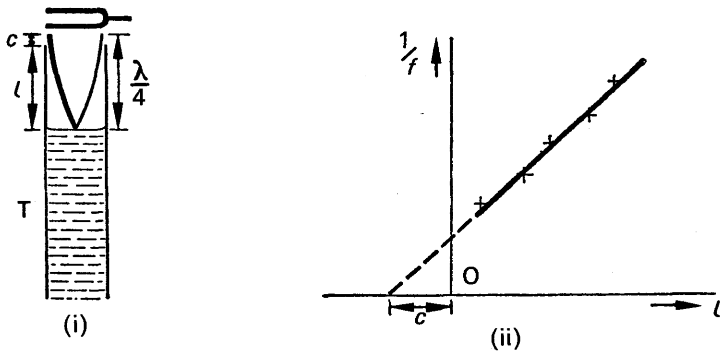


FIG. 27.16. Resonance tube experiment.

shown. If c is the end-correction of the tube (p. 650), and l is the length from the water level to the top of the tube, then

$$l + c = \frac{\lambda}{4} \quad . \quad . \quad . \quad (i)$$

But

$$\lambda = \frac{V}{f},$$

where f is the frequency of the fork and V is the velocity of sound in air.

$$\therefore \quad l + c = \frac{V}{4f} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

If different tuning-forks of known frequency f are taken, and the corresponding values of l obtained when resonance occurs, it follows from equation (ii) that a graph of $1/f$ against l is a straight line, Fig. 27.16 (ii). Now from equation (ii), the gradient of the line is $4/V$; thus V can be determined. Also, the negative intercept of the line on the axis of l is c, from equation (ii); hence the end-correction can be found.

If only one fork is available, and the tube is sufficiently long, another method for V and c can be adopted. In this case the level of the water is lowered further from the position in Fig. 27.16 (i), until resonance is again obtained at a level L_1, Fig. 27.17. Since the stationary wave set up is that shown and the new length to the top from L_1 is l_1, it follows that

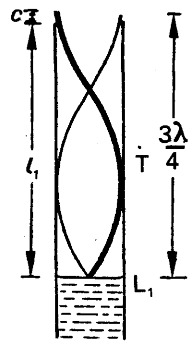


FIG. 27.17. Resonance at new water level.

$$l_1 + c = \frac{3\lambda}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

But $$l + c = \frac{\lambda}{4}, \text{ from (ii).}$$

Subtracting, $$l_1 - l = \frac{\lambda}{2}$$

$$\therefore \quad \lambda = 2(l_1 - l)$$

$$\therefore \quad V = f\lambda = 2f(l_1 - l) \quad . \quad . \quad (3)$$

In this method for V, therefore, the end-correction c is eliminated. The magnitude of c can be found from equations (ii) and (iii). Thus, from (ii),

$$3l + 3c = \frac{3\lambda}{4}$$

But, from (iii), $$l_1 + c = \frac{3\lambda}{4}$$

$$\therefore \quad 3l + 3c = l_1 + c$$

$$\therefore \quad 2c = l_1 - 3l$$

$$\therefore \quad c = \frac{l_1 - 3l}{2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence c can be found from measurements of l_1 and l.

EXAMPLES

1. Describe the natural modes of vibration of the air in an organ pipe closed at one end, and explain what is meant by the term "end-correction". A cylindrical pipe of length 28 cm closed at one end is found to be at resonance when a tuning fork of frequency 864 Hz is sounded near the open end.

Determine the mode of vibration of the air in the pipe, and deduce the value of the end-correction. [Take the velocity of sound in air as 340 m s^{-1}.] (*L.*)

First part. See text.

Second part. Let λ = the wavelength of the sound in the pipe.

Then $$\lambda = \frac{V}{f} = \frac{34000}{864} = 39{\cdot}35 \text{ cm}$$

If the pipe is resonating to its fundamental frequency f_o, the stationary wave in the pipe is that shown in Fig. 27.16 and the wavelength λ_0, is given by $\lambda_o/4 = 28$ cm. Thus $\lambda_o = 112$ cm. Since $\lambda = 39{\cdot}35$ cm, the pipe cannot be sounding its resonant frequency. The first overtone of the pipe is $3f_o$, which corresponds to a wavelength λ_1 given by $3\lambda/4 = 28$ (see Fig. 27.8).

$$\therefore \quad \lambda_1 = \frac{112}{3} = 37\tfrac{1}{3} \text{ cm}$$

Consequently, allowing for the effect of an end-correction, the pipe is sounding its first overtone.

Let c = the end-correction in cm.

Then $$28 + c = \frac{3\lambda_1}{4}$$

But, accurately, $$\lambda_1 = \frac{V}{f} = \frac{34000}{864} = 39{\cdot}35$$

$$\therefore \quad 28 + c = \tfrac{3}{4} \times 39{\cdot}35$$

$$\therefore \quad c = 1{\cdot}5 \text{ cm.}$$

2. Explain the phenomenon of resonance, and illustrate your answer by reference to the resonance-tube experiment. In such an experiment with a resonance tube the first two successive positions of resonance occurred when the lengths of the air columns were 15·4 cm and 48·6 cm respectively. If the velocity of sound in air at the time of the experiment was 34000 cm s^{-1} calculate the frequency of the source employed and the value of the end-correction for the resonance tube. If the air column is further increased in length, what will be the length when the next resonance occurs? (*W.*)

First part. See text.

Second part. Suppose c is the end-correction in cm. Then, from p. 654,

$$48{\cdot}6 + c = \frac{3\lambda}{4} \quad . \quad . \quad . \quad \text{(i)}$$

and $$15{\cdot}4 + c = \frac{\lambda}{4} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Subtracting $$\therefore \quad 33{\cdot}2 = \frac{\lambda}{2}$$

$$\therefore \quad 66{\cdot}4 = \lambda \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$\text{frequency, } f = \frac{V}{\lambda} = \frac{34000}{66{\cdot}4} = 512 \text{ Hz}$$

The end-correction, c, is given by substituting $\lambda = 66{\cdot}4$ in (i). Thus

$$48{\cdot}6 + c = \tfrac{3}{4} \times 66{\cdot}4$$

from which $$c = 1{\cdot}2 \text{ cm.}$$

The next resonance occurs when the total length, a, of the stationary wave set up is $5\lambda/4$. From (iii), $a = \frac{5}{4} \times 66{\cdot}4 = 83{\cdot}0$ cm. Since the end-correction is 1·2 cm,

$$\therefore \quad \text{length of pipe} = 83{\cdot}0 - 1{\cdot}2 = 81{\cdot}8 \text{ cm.}$$

3. Explain, with diagrams, the possible states of vibration of a column of air in (*a*) an open pipe, (*b*) a closed pipe. An open pipe 30 cm long and a closed pipe 23 cm long, both of the same diameter, are each sounding its first overtone, and these are in unison. What is the end-correction of these pipes? (*L.*)

First part. See text.

Second part. Suppose V is the velocity of sound in air, and f is the frequency of the note. The wavelength, λ, is thus V/f.

When the open pipe is sounding its first overtone, the length of the pipe plus end-corrections $= \lambda$.

$$\therefore \quad \frac{V}{f} = 30 + 2c \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

since there are two end-corrections.

When the closed pipe is sounding its first overtone,

$$\frac{3\lambda}{4} = 23 + c$$

$$\therefore \quad \frac{3V}{4f} = 23 + c \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii), it follows that

$$23 + c = \tfrac{3}{4}(30 + 2c)$$

$$\therefore \quad 92 + 4c = 90 + 6c$$

$$\therefore \quad c = 1 \text{ cm.}$$

VIBRATIONS IN STRINGS

If a horizontal rope is fixed at one end, and the other end is moved up and down, a wave travels along the rope. The particles of the rope are then vibrating vertically, and since the wave travels horizontally, this is an example of a *transverse* wave (see p. 584). The waves propagated along the surface of the water when a stone is dropped into it are also transverse waves, as the particles of the water are moving up and down while the wave travels horizontally. A transverse wave is also obtained when a stretched string, such as a violin string, is plucked; and before we can study the vibrations in strings, we require to know the velocity of transverse waves along a string.

Velocity of Transverse Waves Along a Stretched String

Suppose that a transverse wave is travelling along a thin string of length l and mass s under a constant tension T. If we assume that the string has no "stiffness", i.e., that the string is perfectly flexible, the velocity V of the transverse wave along it depends only on the values of T, s, l. The velocity is given by

$$V = \sqrt{\frac{T}{s/l}},$$

$$\text{or} \quad V = \sqrt{\frac{T}{m}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where m is the "mass per unit length" of the string.

When T is in *newtons* and m in *kilogramme per metre*, then V is in *metres per second.*

The formula for V may be partly deduced by the method of dimensions, in which all the quantities concerned are reduced to the fundamental units of mass, M, length, L, and time, T. Suppose that

$$V = kT^x s^y l^z \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where k, x, y, z, are numbers. The dimensions of velocity V are LT^{-1}, the dimensions of tension T, a force, are MLT^{-2}, the dimension of s is M, and the dimension of l is L. As the dimensions on both sides of (i) must be equal, it follows that

$$LT^{-1} = (MLT^{-2})^x(M^y)(L^z)$$

Equating the indices of M, L, T on both sides, we have

for M, $\quad x + y = 0$

for L, $\quad x + z = 1$

for T, $\quad 2x = 1$

$$\therefore \quad x = \tfrac{1}{2}, z = \tfrac{1}{2}, y = -\tfrac{1}{2}$$

Thus, from (i)

$$V = kT^{\frac{1}{2}}s^{-\frac{1}{2}}\, l^{\frac{1}{2}}$$

$$\therefore \quad V = k\sqrt{\frac{Tl}{s}} = k\sqrt{\frac{T}{s/l}}$$

A rigid mathematical treatment shows that $k = 1$, since $V = \sqrt{\frac{T}{s/l}}$. Since s/l is the "mass per unit length" of the string, it follows that

$$V = \sqrt{\frac{T}{m}},$$

where m is the mass per unit length.

Modes of Vibration of Stretched String

If a wire is stretched between two points N, N and is plucked in the middle, a transverse wave travels along the wire and is reflected at the fixed end. A *stationary wave* is thus set up in the wire, and the simplest mode of vibration is one in which the fixed ends of the wire are nodes, N, and the middle is an antinode, A, Fig. 27.18. Since the distance be-

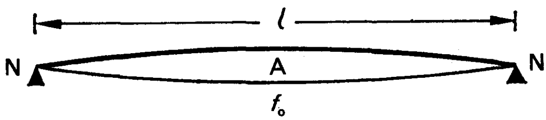


FIG. 27.18. Fundamental of stretched string.

tween consecutive nodes is $\lambda/2$, where λ is the wavelength of the transverse wave in the wire, it follows that

$$l = \frac{\lambda}{2},$$

where l is the length of the wire. Thus $\lambda = 2l$. The frequency f of the vibration is hence given by

$$f = \frac{V}{\lambda} = \frac{V}{2l},$$

where V is the velocity of the transverse wave. But $V = \sqrt{T/m}$, from previous.

$$\therefore \quad f = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

This is the frequency of the *fundamental* note obtained from the string; and if we denote the frequency by the usual symbol f_0, we have

$$f_0 = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad . \quad . \quad . \quad . \quad (6)$$

Overtones of Stretched String

The first overtone f_1 of a string plucked in the middle corresponds to a stationary wave shown in Fig. 27.19, which has nodes at the fixed ends and an antinode in the middle. If λ_1 is the wavelength, it can be seen that

$$l = \frac{3}{2}\lambda_1,$$

$$\text{or } \lambda_1 = \frac{2l}{3}.$$

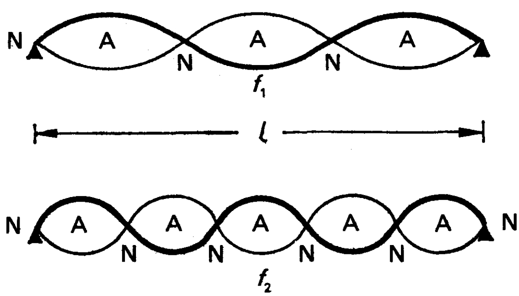


FIG. 27.19. Overtones of stretched string plucked in middle.

The frequency f_1 is thus given by

$$f_1 = \frac{V}{\lambda_1} = \frac{3V}{2l} = \frac{3}{2l}\sqrt{\frac{T}{m}} \quad . \quad . \quad . \quad (i)$$

But the fundamental frequency, f_0, $= \frac{1}{2l}\sqrt{\frac{T}{m}}$, from equation (6).

$$\therefore \quad f_1 = 3f_0$$

The second overtone f_2 of the string when plucked in the middle corresponds to a stationary wave shown in Fig. 27.19. In this case $l = \frac{5}{2}\lambda_2$, where λ_2 is the wavelength.

$$\therefore \quad \lambda_2 = \frac{2l}{5}$$

$$\therefore \quad f_2 = \frac{V}{\lambda_2} = \frac{5V}{2l}$$

where f_2 is the frequency. But $V = \sqrt{T/m}$.

$$\therefore \quad f_2 = \frac{5}{2l}\sqrt{\frac{T}{m}} = 5f_0$$

The overtones are thus $3f_0$, $5f_0$, and so on.

Other notes than those considered above can be obtained by touching or "stopping" the string lightly at its midpoint, for example, so that the latter becomes a node in addition to those at the fixed ends. If the string is plucked one-quarter of the way along it from a fixed end, the simplest stationary wave set up is that illustrated in Fig. 27.20 (i). Thus the wavelength $\lambda = l$, and hence the frequency f is given by

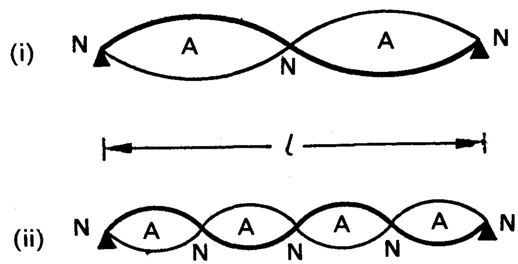


FIG. 27.20. Even harmonics in stretched string.

$$f = \frac{V}{\lambda} = \frac{V}{l} = \frac{1}{l}\sqrt{\frac{T}{m}}$$

$$\therefore \quad f = 2f_0, \text{ since } f_0 = \frac{1}{2l}\sqrt{\frac{T}{m}}.$$

If the string is plucked one-eighth of the way from a fixed end, a stationary wave similar to that in Fig. 27.20 (ii) may be set up. The wavelength, $\lambda' = l/2$, and hence the frequency $f' = \frac{V}{\lambda'} = \frac{2V}{l}$.

$$\therefore \quad f' = \frac{2}{l}\sqrt{\frac{T}{m}} = 4f_0$$

Verification of the Laws of Vibration of a Fixed String. The Sonometer

As we have already shown (p. 658), the frequency of the fundamental of a stretched string is given by $f = \frac{1}{2l}\sqrt{\frac{T}{m}}$, writing f for f_0. It thus, follows that:

(1) $f \propto \frac{1}{l}$ *for a given tension* (T) *and string* (m *constant*).

(2) $f \propto \sqrt{T}$ *for a given length* (l) *and string* (m *constant*).

(3) $f \propto \frac{1}{\sqrt{m}}$ *for a given length* (l) *and tension* (T).

These are known as the "laws of vibration of a fixed string", first completely given by MERSENNE in 1636, and the *sonometer*, or *monochord*, was designed to verify them.

The sonometer consists of a hollow wooden box Q, with a thin horizontal wire attached to A on the top of it, Fig. 27.21. The wire passes

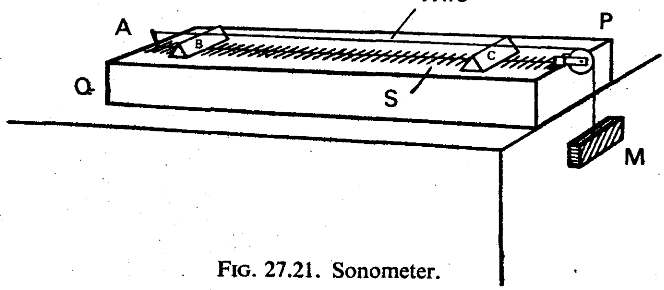


FIG. 27.21. Sonometer.

over a grooved wheel P, and is kept taut by a mass M hanging down at the other end. Wooden bridges, B, C, can be placed beneath the wire so that a definite length of wire is obtained, and the length of wire can be varied by moving one of the bridges. The length of wire between B, C can be read from a fixed horizontal scale S, graduated in centimetres, on the box below the wire.

(1) *To verify* $f \propto 1/l$ *for a given tension* (T) *and mass per unit length* (m), the mass M is kept constant so that the tension, T, in the wire is constant. The length, l, of the wire between B, C is varied by moving C until the note obtained by plucking BC in the middle is the same as that produced by a sounding tuning-fork of known frequency f. If the observer lacks a musical ear, the "tuning" can be recognised by listening for beats when the wire and the tuning-fork are both sounding, as in this case the frequencies of the two notes are nearly equal (p. 620). Alternatively, a small piece of paper in the form of an inverted V can be placed on the middle of the wire, and the end of the sounding tuning-fork then placed on the sonometer box. The vibrations of the fork are then transmitted through the box to the wire, which vibrates in resonance with the fork if its length is "tuned" to the note. The paper will then vibrate considerably and may be thrown off the wire.

Different tuning-forks of known frequency f are taken, and the lengths, l, of the wire are observed when they are tuned to the corresponding note. A graph of f against $1/l$ is then plotted, and is found to be a straight line within the limits of experimental error. Thus $f \propto 1/l$ for a given tension and mass per unit length of wire.

(2) *To verify $f \propto \sqrt{T}$ for a given length and mass per unit length*, the length BC between the bridges is kept fixed, so that the length of wire is constant, and the mass M is varied to alter the tension. The experimental difficulty to be overcome is how to find the frequency f of the note produced when the wire between B, C is plucked in the middle. For this purpose a second wire, fixed to R, S on the sonometer, is utilised, usually with a weight (not shown) attached to one end to keep the tension constant, Fig. 27.22. This wire has bridges P, N beneath it, and N is moved until the note from the wire between P, N is the same as the note from the wire between B, C. Now the tension in PN is constant as the wire is fixed to R, S. Thus, since frequency, f, $\propto 1/l$ for a given tension and wire, the frequency of the note from BC is proportional to $1/l$, where l is the length of PN.

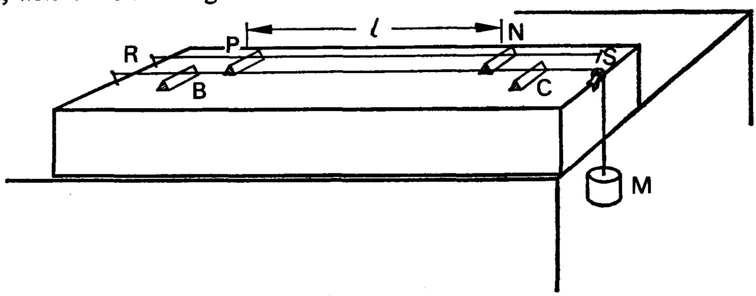


FIG. 27.22. Verification of $f \propto \sqrt{T}$.

If a different mass is attached to the end of the wire BC, the tension in the wire is altered. Keeping BC fixed, the bridge N is moved until the note from PN is the same as that obtained from BC, and the length PN (l) is again noted. Then, as before, the frequency of the new note in BC is proportional to $1/l$. By altering the mass M, and observing the corresponding length l, a graph of $1/l$ can be plotted against $\sqrt{T}$, where T is the weight of M. The graph is a straight line, within the limits of experimental error, and hence $1/l \propto \sqrt{T}$. Thus $f \propto \sqrt{T}$ for a given length of wire and mass per unit length.

(3) *To verify $f \propto 1/\sqrt{m}$ for a given length and tension*, wires of different material are connected to B, C, and the same mass M and the same length BC are taken. The frequency, f, of the note obtained from BC is again found by using the second wire RS in the way already described. The mass per unit length, m, is the mass per metre length of wire, and is given by $\pi r^2 \rho$ kg m^{-1}, where r is the radius of the wire in m and ρ is its density in kg m^{-3}, as ($\pi r^2 \times 1$) m^3 is the volume of 1 m of the wire. Since $f \propto 1/l$, where l is the length on the second wire, a graph of $1/l$ against $1/\sqrt{m}$ should be a straight line; thus a graph of l against $\sqrt{m}$ should be a straight line if $f \propto 1\sqrt{m}$ for a given length and mass per unit length. Experiment shows this is the case.

Melde's Experiment on Vibrations in Stretched String

MELDE gave a striking demonstration of the stationary wave set up in a vibrating string. He used a light thread with one end attached to the prong P of an electrically-maintained tuning-fork, and the other end connected to a weight W after passing over a grooved wheel Q, Fig. 27.23. With the vibrations of the prong perpendicular to the length

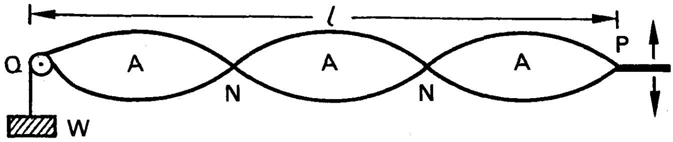


FIG. 27.23. Melde's experiment.

of the thread a number of "loops" can be observed, which are due to the rapid movement of the thread as the stationary transverse wave passes along it. The nodes of the stationary wave are at Q and at P, as the amplitude of vibration of the prong is small, and the number n of the loops depends on the frequency f of the fork, the length l of PQ, the tension T in the thread, and its mass per unit length, m.

The length of a loop $= l/n$. But this is the distance between consecutive nodes, which is $\lambda/2$, where λ is the wavelength of the stationary wave in the thread.

$$\therefore \quad \frac{\lambda}{2} = \frac{l}{n}$$

$$\therefore \quad \lambda = \frac{2l}{n}$$

$$\therefore \quad f = \frac{V}{\lambda} = \frac{n}{2l}V,$$

where V is the velocity of the transverse wave. The frequency of the transverse wave is the same as that of the tuning-fork.

But $$V = \sqrt{\frac{T}{m}} \text{ (p. 657)}$$

$$\therefore \quad f = \frac{n}{2l}\sqrt{\frac{T}{m}}, \qquad (7)$$

$$\therefore \quad n\sqrt{T} = \text{constant},$$

if f, T, m are kept constant. In this case, therefore,

$$n = \frac{\text{constant}}{\sqrt{T}}$$

$$\text{or } n^2 \propto \frac{1}{T} \qquad (8)$$

This relation can be verified by varying the tension T in the thread, and obtaining a corresponding whole number, n, of loops. A graph of n^2 against $1/T$ is then plotted, and is a straight line passing through the origin.

When the prong P of the tuning-fork is vibrating in the same direction as the length of string, the latter moves from position 3 to position 2 as the prong moves from one end a of an oscillation to the other end b,

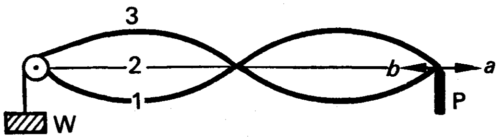


FIG. 27.24. Melde's experiment.

Fig. 27.24. As the string continues from position 2 to position 1, the prong moves back from b to a to complete 1 cycle of oscillation. The fork thus goes through one complete cycle in the same time as the particles of the string go through half a cycle, and hence the frequency of the transverse wave is *half* the frequency f of the fork, unlike the first case considered, Fig. 27.24. Instead of equation (7), we now have

$$\frac{f}{2}=\frac{n}{2l}\sqrt{\frac{T}{m}}$$

$$\text{or} \qquad n=fl/\sqrt{T/m},$$

and hence with the same tension, thread, and length l, the number of loops n is half that obtained previously (p. 662).

Measurement of the Frequency of A.C. Mains

The frequency of the alternating current (A.C.) mains can be determined with the aid of a sonometer wire. The alternating current is passed into the wire MP, and the poles N, S of a powerful magnet are placed on either side of the wire so that the magnetic field due to it is perpendicular to the wire, Fig. 27.25. As a result of the magnetic effect of the current, a force acts on the wire which is perpendicular to the directions of both the magnetic field and the current, and hence the wire is subjected to a tranverse force. If the current is an alternating one of 50 Hz, the magnitude of the force varies at the rate of 50 Hz. By adjusting the tension in the sonometer wire, whose magnitude is read on the spring-balance A, a position can be reached when the wire is seen to be vibrating through a large amplitude; in this case the wire is *resonating* to the applied force, Fig. 27.25.

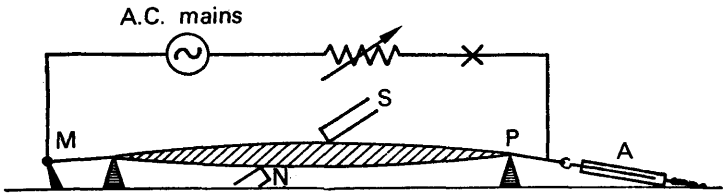


FIG. 27.25. Measurement of frequency of A.C. mains.

The length l of wire between the bridges is now measured, and the tension T and the mass per unit length, m, are also found. The frequency f of the alternating current is then given by

$$f=\frac{1}{2l}\sqrt{\frac{T}{m}}.$$

Velocity of Longitudinal Waves in Wires

If a sonometer wire is stroked along its length by a rosined cloth, a high-pitched note is obtained. This note is due to *longitudinal* vibrations in the wire, and must be clearly distinguished from the note produced when the wire is plucked, which sets up *transverse* vibrations of the wire and a corresponding transverse wave. As we saw on p. 594, the velocity V of a longitudinal wave in a medium is

$$V = \sqrt{\frac{E}{\rho}},$$

where E is Young's modulus for the wire and ρ is its density. The wavelength, λ, of the longitudinal wave is $2l$, where l is the length of the wire, since a stationary longitudinal wave is set up. Thus the frequency f of the note is given by

$$f = \frac{V}{\lambda} = \frac{1}{2l}\sqrt{\frac{E}{\rho}}.$$

The frequency of the note may be obtained with the aid of an audio oscillator, and thus the velocity of sound in the wire, or its Young's modulus, can be calculated.

EXAMPLES

1. Explain the meaning of the term *resonance*, giving in illustration two methods of obtaining resonance between the stretched string of a sonometer and a tuning-fork of fixed frequency. A sonometer wire of length 76 cm is maintained under a tension of value 4 kgf and an alternating current is passed through the wire. A horse-shoe magnet is placed with its poles above and below the wire at its midpoint, and the resulting forces set the wire in resonant vibration. If the density of the material of the wire is 8800 kg m^{-3} and the diameter of the wire is 1 mm, what is the frequency of the alternating current? (*L.*)

First part. See text.

Second part. The wire is set into resonant vibration when the frequency of the alternating current is equal to its natural frequency, f.

Now $$f = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \text{. . . . . . (i)}$$

where $l = 0{\cdot}76$ m, $T = 4 \times 9{\cdot}8$ newton, and m = mass per metre in kg m^{-1}.

Also, mass of 1 metre = volume $\times$ density

$= \pi r^2 \times 1 \times 8800$ kg,

where radius r of wire $= \frac{1}{2}$ mm $= 0{\cdot}5 \times 10^{-3}$ m

From (i), $$\therefore \quad f = \frac{1}{2 \times 0{\cdot}76}\sqrt{\frac{4 \times 9{\cdot}8}{\pi \times 0{\cdot}5^2 \times 10^{-6} \times 1 \times 8800}}$$

$= 49{\cdot}6$ Hz.

2. What data would be required in order to predict the frequency of the note emitted by a stretched wire (*a*) when it is plucked, (*b*) when it is stroked along its length? A weight is hung on the wire of a vertical sonometer. When the vibrating length of the wire is adjusted to 80 cm the note it emits when plucked is in tune with a standard fork. On adding a further weight of 100 g the vibrating length has to be altered by 1 cm in order to restore the tuning. What is the initial weight on the wire? (*L.*)

First part. When the wire is plucked the vibrations of the particles produce transverse waves, and the frequency of the note is given by $f = \frac{1}{2l}\sqrt{\frac{T}{m}}$. When the wire is stroked along its length, the vibrations of the particles produce a *longitudinal* wave, and the velocity of the wave is given by $V = \sqrt{E/\rho}$, where E is Young's modulus for the wire and ρ is its density. The frequency in the latter case thus depends on the magnitudes of E and ρ, as well as on the length of the wire.

Second part. Let W = the initial weight on the wire in kgf,
and f = the frequency of the fork

Since
$$= \frac{1}{2l}\sqrt{\frac{T}{m}}$$

we have
$$f = \frac{1}{2 \times 0{\cdot}80}\sqrt{\frac{Wg}{m}} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

When a weight of 0·1 kgf is added, the frequency increases since the tension increases. The new length of the wire = 0·81 m.

$$\therefore \quad f = \frac{1}{2 \times 0{\cdot}81}\sqrt{\frac{(W + 0{\cdot}1)g}{m}} \quad . \quad . \quad \text{(ii)}$$

From (i) and (ii), it follows that

$$\frac{1}{1{\cdot}60}\sqrt{\frac{Wg}{m}} = \frac{1}{1{\cdot}62}\sqrt{\frac{(W + 0{\cdot}1)g}{m}}$$

$$\therefore \quad 162^2\, W = 160^2\, (W + 0{\cdot}1)$$

$$W = \frac{160^2 \times 0{\cdot}1}{162^2 - 160^2} = 4 \text{ kgf (approx.)}$$

VIBRATIONS IN RODS

Sound waves travel through liquids and solids, as well as through gases, and in the nineteenth century an experiment to measure the velocity of sound in iron was carried out by tapping one end of a very long iron tube. The speed of sound in iron is much greater than in air, and the sound through the iron thus arrived at the other end of the pipe before the sound transmitted through the air. From a knowledge of the interval between the sounds, the length of the pipe, and the velocity of sound in air, the velocity of sound in iron was determined. More accurate methods were soon forthcoming for the velocity of sound in substances such as iron, wood, and glass, and they depend mainly on the formation of stationary waves in rods of these materials.

Consider a rod AA fixed by a vice B at its mid-point N, Fig. 27.26. If the rod is stroked along its length by a rosined cloth, a stationary longitudinal wave is set up in the rod on account of reflection at its ends, and a high-pitched note is obtained. Since the mid-point of the rod is fixed, this is a node, N, of the stationary wave; and since the ends

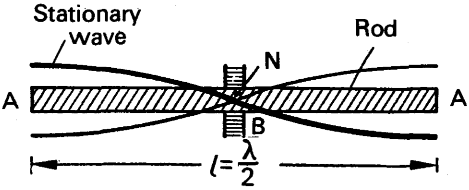


FIG. 27.26. Stationary wave in rod.

of the rod are free, these are antinodes, A. Thus the length l of the rod is equal to half the wavelength, $\lambda/2$, of the wave in the rod, and hence $\lambda = 2l$. Thus the velocity of the sound in the rod, $V = f\lambda = f \times 2l$, where f is the frequency of the note from the rod.

Kundt's Tube

About 1868, KUNDT devised a simple method of showing the stationary waves in air or any other gas. He used a closed tube T containing the gas, and sprinkled some dry lycopodium powder, or cork dust, along the entire length, Fig. 27.27. A rod AE, clamped at its mid-point, is placed with one end projecting into T, and a disc E is attached at this end so that it just clears the sides of the tube, Fig. 27.27. When the

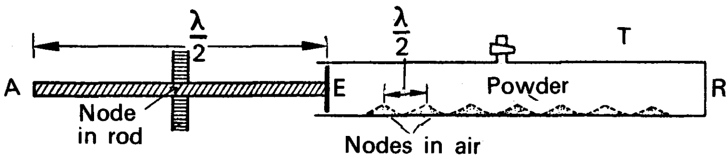


FIG. 27.27. Kundt's tube.

rod is stroked at A by a rosined cloth in the direction EA, the rod vibrates longitudinally and a high-pitched note can be heard. The end E acts as a vibrating source of the same frequency, and a sound wave thus travels through the air in T and is reflected at the fixed end R. If the rod is moved so that the position of E alters, a position can be found when the stationary wave in the air in T causes the lycopodium powder to become violently agitated. The powder then settles into definite small heaps at the nodes, which are the positions of permanent rest of the stationary wave, and the distance between consecutive nodes can best be found by measuring the distance between several of them and dividing by the appropriate number.

Determination of Velocity of Sound in a Rod

Kundt's tube can be used to determine the velocity of sound, V_r, in the rod. Suppose the length of the rod is l: then $\lambda/2 = l$, or $\lambda = 2l$, where λ is the wavelength of the sound wave *in the rod* (p. 643). Thus the frequency of the high-pitched note obtained from the rod is given by

$$f = \frac{V_r}{\lambda} = \frac{V_r}{2l} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

If l_1 is the distance between consecutive nodes of the stationary wave in the air, we have $\lambda_1/2 = l_1$, where λ_1 is the wavelength of the sound wave *in the air*. Thus $\lambda_1 = 2l_1$, and hence the frequency of the wave, which is also f, is given by

$$f = \frac{V_a}{\lambda} = \frac{V_a}{2l_1}, \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

where V_a is the velocity of sound in air. From (i) and (ii) it follows that

$$\frac{V_r}{2l} = \frac{V_a}{2l_1}$$

$$\therefore \quad V_r = \frac{l}{l_1} V_a \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus knowing V_a, l, l_1, the velocity of sound in the rod, V_r, can be calculated. By using glass, brass, copper, steel and other substances in the form of a rod, the velocity of sound in these media have been determined. Kundt also used liquids in the tube T instead of air, and employed fine iron filings instead of lycopodium powder to detect the nodes in the liquid. In this way he determined the velocity of sound in liquids.

Determination of Young's Modulus of a Rod

On p. 622, it was shown that the velocity of sound, V, in a medium is always given by

$$V = \sqrt{\frac{E}{\rho}},$$

where E is the appropriate modulus of elasticity of the medium and ρ is its density. In the case of a rod undergoing longitudinal vibrations, as in Kundt's tube experiment, E is Young's modulus (see p. 594). Thus if V_r is the velocity of sound in the rod,

$$V_r = \sqrt{\frac{E}{\rho}},$$

and
$$\therefore \quad E = V_r^2 \rho \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since V_r is obtained by the method explained above, and ρ can be obtained from tables, it follows that E can be calculated.

Determination of Velocity of Sound in a Gas

If the air in Kundt's tube T is replaced by some other gas, and the rod stroked, the average distance l' between the piles of dust in T is the distance between consecutive nodes of the stationary wave in the gas. The wavelength, λ_g, in the gas is thus $2l'$, and the frequency f is given by

$$f = \frac{V_g}{\lambda_g} = \frac{V_g}{2l'},$$

where V_g is the velocity of sound in the gas. But the wavelength, λ, of the wave in the rod $= 2l$, where l is the length of the rod (p. 666); hence f is also given by

$$f = \frac{V_r}{\lambda} = \frac{V_r}{2l}$$

$$\therefore \quad \frac{V_g}{2l'} = \frac{V_r}{2l}$$

$$\therefore \quad V_g = \frac{l'}{l} V_r \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Knowing l', l, and V_r, the latter obtained from a previous experiment (p. 667), the velocity of sound in a gas, V_g, can be calculated. The velocity of sound in a gas can also be found by the more direct method described below.

Determination of Ratio of Specific Heat Capacities of a Gas, and its Molecular Structure

The velocity of sound in a gas, V_g, is given by

$$V_g = \sqrt{\frac{\gamma p}{\rho}},$$

where γ is the ratio (c_p/c_v) of the two principal specific heat capacities of the gas, p is its pressure, and ρ is its density. See p. 624. Thus

$$\gamma = \frac{V_g^2 \rho}{p} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Now it has already been shown that V_g can be found; and knowing ρ and p, γ can be calculated. The determination of γ is one of the most important applications of Kundt's tube, as kinetic theory shows that $\gamma = 1{\cdot}66$ for a monatomic gas and $1{\cdot}40$ for a diatomic gas. Thus Kundt's tube provides valuable information about the molecular structure of a gas. When RAMSEY isolated the hitherto-unobtainable argon from the air, Lord Rayleigh in 1895 suggested a Kundt's tube experiment for finding the ratio γ, of its specific heats. It was then discovered that γ was about $1{\cdot}65$, showing that argon was a monatomic gas. The dissociation of the molecules of a gas at high temperatures has been investigated by containing it in Kundt's tube surrounded by a furnace, and measuring the magnitude of γ when the temperature was changed.

Comparison of Velocities of Sound in Gases by Kundt's Tube

The ratio of the velocities of sound in two gases can be found from a Kundt's tube experiment. The two gases, air and carbon dioxide for example, are contained in tubes A, B respectively, into which the ends of a metal rod R project, Fig. 27.28. The middle of the rod is clamped. By stroking the rod, and adjusting the positions of the movable discs Y, X in turn, lycopodium powder in each tube can be made to settle into heaps at the various nodes. The average distances, d_a, d_b, between successive nodes in A, B respectively are then measured.

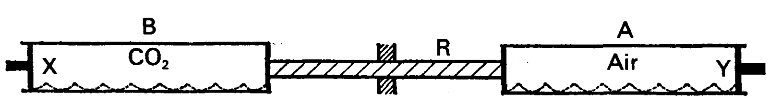


FIG. 27.28. Comparison of velocities of sound in gases.

The frequency f of the sound wave in A, B is the same, being the frequency of the note obtained from R. Since $f = V/\lambda$, it follows that

$$\frac{V_g}{\lambda_g} = \frac{V_a}{\lambda_a}, \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where V_g, V_a are the velocities of sound in carbon dioxide and air respectively, and λ_g, λ_a are the corresponding wavelengths.

Now $$\frac{\lambda_g}{\lambda_a} = \frac{d_b}{d_a} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

since the distance between successive nodes is half a wavelength From (i),

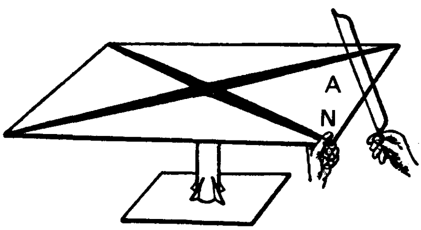
$$\frac{V_g}{V_a} = \frac{\lambda_g}{\lambda_a}$$

$$\therefore \quad \frac{V_g}{V_a} = \frac{d_b}{d_a} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

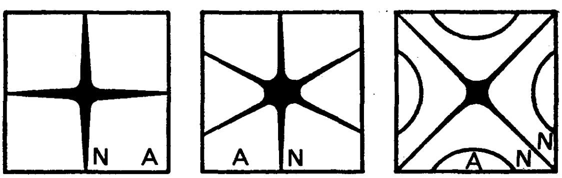
The two velocities can thus be compared as d_b, d_a are known; and if the velocity of sound, V_a, in air is known, the velocity in carbon dioxide can be calculated.

Vibrations in Plates. Chladni's Figures

We have already studied the different modes of vibration of the air in a pipe, the particles of a string, and the particles of a rod. About 1790 CHLADNI examined the vibrations of a glass *plate* by sprinkling sand on it. If the plate on a stand is gripped firmly at the corner N and bowed in the middle A of one side, the particles arrange themselves into a symmetrical pattern which shows the nodes of the stationary wave in the plate, Fig. 27.29 (i). By gripping the plate firmly at other points N, thus making a node at these points, and bowing at A, a series



(i)



(ii)

FIG. 27.29. Chladni's figures.

of different patterns can be obtained. These are known as *Chladni's figures*, Fig. 27.29 (ii). Each pattern corresponds to a particular mode of vibration. These modes are not harmonically related in frequency, unlike the case of the vibration of air in pipes and the vibration of strings.

EXAMPLES

1. Describe and explain the way in which a Kundt tube may be used to determine the ratio of the specific heats of a gas. A Kundt tube is excited by a brass rod 150 cm long and the distance between successive nodes in the tube is 13·6 cm; what is the ratio of the velocity of sound in brass to that in air? (*L.*)

First part. See text.

Second part. Since both ends of the rod are successive antinodes, the wavelength λ_1 in the rod $= 2 \times 150 = 300$ cm. The wavelength λ_2 in the air $= 2 \times 13{\cdot}6 = 27{\cdot}2$ cm.

The frequency f of the note in the rod and the air is the same.

$$\therefore \quad f = \frac{V_1}{\lambda_1} = \frac{V_2}{\lambda_2}$$

where V_1, V_2 are the velocities of sound in the rod and in the air.

$$\therefore \quad \frac{V_1}{V_2} = \frac{\lambda_1}{\lambda_2} = \frac{300}{27{\cdot}2} = 11{\cdot}0$$

2. Describe the dust tube experiment. How may it be used to compare the velocities of sound in different gases? The fundamental frequency of longitudinal vibration of a rod clamped at its centre is 1500 Hz. If the mass of the rod is 96·0 g, find the increase in its total length produced by a stretching force of 10 kgf (*L.*)

First part. The dust tube is Kundt's tube. See p. 666.

Second part. The wavelength of the wave in the rod $= 2l$, where l metre is its length, since the ends are antinodes. The velocity V, of the wave is given by

$$V = f\lambda = 1500 \times 2l = 3000\,l \quad . \quad . \quad . \quad \text{(i)}$$

Since the vibrations of the rod are longitudinal,

$$V = \sqrt{\frac{E}{\rho}}.$$

E is Young's modulus in N m^{-2} and ρ is the density of the rod in kg m^{-3}

$$\therefore \quad V = \sqrt{\frac{E}{0{\cdot}096/v}} = \sqrt{\frac{Ev}{0{\cdot}096}} \quad . \quad . \quad \text{(ii)}$$

where v is the volume of the rod in metre^3.

From (i) and (ii),

$$\sqrt{\frac{Ev}{0{\cdot}096}} = 3000\,l$$

$$\therefore \quad \frac{Ev}{l^2} = 0{\cdot}096 \times 3000^2$$

$$\therefore \quad \frac{EA}{l} = 0{\cdot}096 \times 3000^2 \quad . \quad . \quad . \quad \text{(iii)}$$

since $v = Al$, where A is the area of cross-section of the rod. Now if x is the increase in length produced by 10 kgf, it follows from the definition of E that

$$\text{force} = \frac{EAx}{l} = 10 \times 9{\cdot}8 \text{ newtons}$$

$$\therefore \quad \frac{EA}{l} = \frac{10 \times 9{\cdot}8}{x}$$

$$\text{From (iii)}, \therefore 0{\cdot}096 \times 3{,}000^2 = \frac{10 \times 9{\cdot}8}{x}$$

$$\therefore \quad x = \frac{10 \times 9{\cdot}8}{0{\cdot}096 \times 3000^2} = 1{\cdot}1 \times 10^{-4} \text{ metre.}$$

EXERCISES 27

Pipes

1. Write down in terms of wavelength, λ, the distance between (i) consecutive nodes, (ii) a node and an adjacent antinode, (iii) consecutive antinodes. Find the frequency of the fundamental of a closed pipe 15 cm long if the velocity of sound in air is 340 m s^{-1}.

2. Discuss what is meant by the statement that *sound is a wave motion*. Use the example of the passage of a sound wave through air to explain the terms wavelength (λ), frequency (f), and velocity (v) of a wave. Show that $v = f\lambda$.

Explain the increase in loudness (or 'resonance') which occurs when a sounding tuning-fork is held near the open end of an organ pipe when the length of the pipe has certain values, the other end of the pipe being closed. Find the shortest length of such a pipe which resonates with a 440 Hz tuning-fork, neglecting end corrections. (Velocity of sound in air = 350 m s^{-1}.) (*O. & C.*)

3. What are the chief characteristics of a progressive wave motion? Give your reasons for believing that sound is propagated through the atmosphere as a longitudinal wave motion, and find an expression relating the velocity, the frequency, and the wavelength.

Neglecting end effects, find the lengths of (*a*) a closed organ pipe, and (*b*) an open organ pipe, each of which emits a fundamental note of frequency 256 Hz. (Take the speed of sound in air to be 330 m s^{-1}.) (*O.*)

4. (*a*) Explain in terms of the properties of a gas, but without attempting mathematical treatment, how the vibration of a sound source, such as a loudspeaker diaphragm, can be transmitted through the air around it.

Explain, also, the reflection which occurs when the vibration reaches a fixed barrier, such as a wall.

(*b*) Plane, simple harmonic, progressive sound waves of wavelength 1·2 m and speed 348 m s^{-1}, are incident normally on a plane surface which is a perfect reflector of sound. What statements can be made about the amplitude of vibration and about air pressure changes at points distant (i) 30 cm, (ii) 60 cm, (iii) 90 cm, (iv) 10 cm from the reflector? Justify your answers. (*O. & C.*)

5. Describe the motion of the air in a tube closed at one end and vibrating in its fundamental mode. An observer (*a*) holds a vibrating tuning-fork over the open end of a tube which resounds to it, (*b*) blows lightly across the mouth of the tube. Describe and explain the difference in the quality of the notes that he hears.

A uniform tube, 60·0 cm long, stands vertically with its lower end dipping into water. When the length above water is 14·8 cm, and again when it is 48·0 cm, the tube resounds to a vibrating tuning fork of frequency 512 Hz. Find the lowest frequency to which the tube will resound when it is open at both ends. (*L.*)

6. Discuss the factors which determine the pitch of the note given by a 'closed' pipe. Explain why the fundamental frequency and the quality of the note from a 'closed' pipe differ from those of the note given under similar conditions by a pipe of the same length which is open at both ends. (*N.*)

7. What is meant by (*a*) *a stationary wave motion* and (*b*) *a node*?

Describe how the phenomenon of resonance may be demonstrated using a loudspeaker, a source of alternating voltage of variable frequency and a suitable tube open at one end and closed at the other. Explain how resonance occurs in the arrangement you describe, draw a diagram showing the position of the nodes in the tube in a typical case of resonance and state clearly the meaning of the diagram. How would you demonstrate the position of the nodes experimentally? *(O. & C.)*

8. Explain the meaning of (*a*) the *end correction* of a resonance tube, (*b*) *beats*. Establish a formula for the frequency of beats in terms of the superimposed frequencies.

A closed resonance tube with an end correction of 0·60 cm is made to sound its fundamental note on a day when the air temperature is 17° C. It is found to be in unison with a siren whose disc, which has 12 holes, is revolving at a rate of 43·0 rev s^{-1}. Calculate (i) the length of the tube, (ii) the frequency of the beats produced if the experiment is repeated on a day when the air temperature has fallen to 12° C, the rate of revolution of the siren's disc being unaltered. (The velocity of sound in air at 0° C may be taken as 331·5 m s^{-1}.) (*L.*)

9. Distinguish between the formation of an echo and the formation of a stationary sound wave by reflection, explaining the general circumstances in which each is produced.

Describe an experiment in which the velocity of sound in air may be determined by observations on stationary waves.

An organ pipe is sounded with a tuning-fork of frequency 256 Hz. When the air in the pipe is at a temperature of 15° C, 23 beats are heard in 10 seconds; when the tuning-fork is loaded with a small piece of wax, the beat frequency is found to decrease. What change of temperature of the air in the pipe is necessary to bring the pipe and the unloaded fork into unison? (*C.*)

10. Describe and give the theory of one experiment in each instance by which the velocity of sound may be determined, (*a*) in free air, (*b*) in the air in a resonance tube.

What effect, if any, do the following factors have on the velocity of sound in free air; frequency of the vibrations; temperature of the air; atmospheric pressure; humidity?

State the relationship between this velocity and temperature. (*L.*)

Strings. Rods

11. What is meant by a *wave motion*? Define the terms *wavelength* and *frequency* and derive the relationship between them.

Given that the velocity v of transverse waves along a stretched string is related to the tension F and the mass m per unit length by the equation

$$v = \sqrt{\frac{F}{m}},$$

derive an expression for the natural frequencies of a string of length l when fixed at both ends.

Explain how the vibration of a string in a musical instrument produces sound and how this sound reaches the ear. Discuss the factors which determine the quality of the sound heard by the listener. *(O. & C.)*

12. Distinguish between a *progressive* wave and a *stationary* wave. Explain in detail how you would use a sonometer to establish the relation between the fundamental frequency of a stretched wire and (*a*) its length, (*b*) its tension. You may assume a set of standard tuning-forks and a set of weights in steps of half a kilogram to be available.

A pianoforte wire having a diameter of 0·90 mm is replaced by another wire of the same material but with diameter 0·93 mm. If the tension of the wire is the same as before, what is the percentage change in the frequency of the fundamental note? What percentage change in the tension would be necessary to restore the original frequency? (*L.*)

13. What is meant by (*a*) a forced vibration, (*b*) resonance? Give an example of each from (i) mechanics, (ii) sound.

Using the same axes sketch graphs showing how the amplitude of a forced vibration depends upon the frequency of the applied force when the damping of the system is (*a*) light, (*b*) heavy. Point out any special features of the graphs.

A sonometer wire is stretched by hanging a metal cylinder of density 8000 kg m^{-3}. at the end of the wire. A fundamental note of frequency 256 Hz is sounded when the wire is plucked.

Calculate the frequency of vibration of the same length of wire when a vessel of water is placed so that the cylinder is totally immersed. (*N.*)

14. Describe an experiment to determine the velocity of sound in a gas, e.g. nitrogen. How would you expect the velocity to be affected by (*a*) temperature, (*b*) pressure and (*c*) humidity? Give reasons for your answers.

What information about the nature of a gas can be obtained from a measurement of the velocity of sound in that gas, the pressure and density being known? (*L.*)

15. Describe experiments to illustrate the differences between (*a*) *transverse* waves, (*b*) *longitudinal* waves, (*c*) *progressive* waves and (*d*) *stationary* waves? To which classes belong (i) the vibrations of a violin string, (ii) the sound waves emitted by the violin into the surrounding air?

A wire whose mass per unit length is 10^{-3} kg m^{-1} is stretched by a load of 4 kg over the two bridges of a sonometer 1 m apart. If it is struck at its middle point, what will be (*a*) the wavelength of its subsequent fundamental vibrations, (*b*) the fundamental frequency of the note emitted? If the wire were struck at a point near one bridge what further frequencies might be heard? (Do not derive standard formulae.) (Assume $g = 10$ m s^{-2}.) *(O. & C.)*

16. A uniform wire vibrates transversely in its fundamental mode. On what factors, other than the length does the frequency of vibration depend, and what is the form of the dependence for each factor?

Describe the experiment you would perform to verify the form of dependence for *one* factor.

A wire of diameter 0·040 cm and made of steel of density 8000 kg m^{-3} is under constant tension of 8·0 kgf. A fixed length of 50 cm is set in transverse vibration. How would you cause the vibration of frequency about 840 Hz to predominate in intensity? (*N.*)

17. (*a*) The velocity of sound in air being known, describe how Young's modulus for brass may be found using Kundt's tube. (*b*) Discuss how the frequency of a note heard by an observer is affected by movement of (i) the source, (ii) the observer along the line joining source and observer. (*L.*)

18. Give an expression for the velocity of a transverse wave along a thin flexible string and show that it is dimensionally correct. Explain how reflexion may give rise to transverse *standing waves* on a stretched string and use the expression for the velocity to derive the frequency of the fundamental mode of vibration.

A steel wire of length 40·0 cm and diameter 0·0250 cm vibrates transversely in unison with a tube, open at each end and of effective length 60·0 cm, when each is sounding its fundamental note. The air temperature is 27° C. Find in kilograms force the tension in the wire. (Assume that the velocity of sound in air at 0° C is 331 m s^{-1} and the density of steel is 7800 kg m^{-3}.) (*L.*)

19. It may be shown theoretically that the frequency f of the fundamental note emitted as the result of the transverse vibration of a stretched wire is given by $f = kT^{\frac{1}{2}}l^{-1}$, where T is the tension in the wire and l its length, k being a constant for a given wire. Describe the experiments you would perform to check this relation. assuming that tuning-forks of known frequencies covering a range of one octave are available.

A brass wire is tuned so that its fundamental frequency is 100 Hz, and a horse-shoe magnet is placed so that the mid-point of the wire lies between its poles. On passing an alternating current through the wire it vibrates, the amplitude of vibration depending upon the frequency of the current. Explain this, and show for what frequencies the vibration will be particularly strong. (*C.*)

20. Explain why the velocity of sound in a gas depends upon the ratio of its principal specific heats.

Give a detailed account of a method of determining this ratio for carbon dioxide gas at atmospheric temperature, assuming that the ratio for air is known. Give the theory of the experiment. (*W.*)