chapter two

Circular motion. S.H.M. Gravitation

Angular Velocity

IN the previous chapter we discussed the motion of an object moving in a straight line. There are numerous cases of objects moving in a *curve* about some fixed point. The earth and the moon revolve continuously round the sun, for example, and the rim of the balance-wheel of a watch moves to-and-fro in a circular path about the fixed axis of the wheel. In this chapter we shall study the motion of an object moving in a circle with a *uniform speed* round a fixed point O as centre, Fig. 2.1.

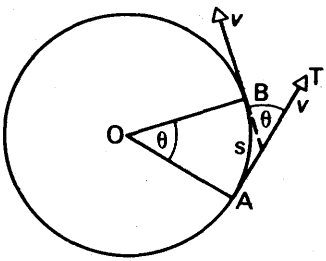

FIG. 2.1 Circular motion

If the object moves from A to B so that the radius OA moves through an angle θ, its *angular velocity*, ω, about O is defined as the *change of the angle per second*. Thus if t is the time taken by the object to move from A to B,

$$\omega = \frac{\theta}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Angular velocity is usually expressed in 'radian per second' (rad s^{-1}). From (1),

$$\theta = \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is analogous to the formula 'distance = uniform velocity × time' for motion in a straight line. It will be noted that the time T to describe the circle once, known as the *period* of the motion, is given by

$$T = \frac{2\pi}{\omega}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since 2π radians = 360° by definition.

If s is the length of the arc AB, then $s/r = \theta$, by definition of an angle in radians.

$$\therefore s = r\theta.$$

Dividing by t, the time taken to move from A to B,

$$\therefore \frac{s}{t} = r\frac{\theta}{t}.$$

But s/t = the *velocity*, v, of the rotating object, and θ/t is the angular velocity.

$$\therefore v = r\omega \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Acceleration in a circle

When a stone is attached to a string and whirled round at constant speed in a circle, one can feel the force in the string needed to keep the stone moving. The presence of the force, called a *centripetal force,* implies that the stone has an acceleration. And since the force acts towards the centre of the circle, the direction of the acceleration, which is a vector quantity, is also towards the centre.

To obtain an expression for the acceleration towards the centre, consider an object moving with a constant speed v round a circle of radius r. Fig. 2.2 (i). At A, its *velocity* v_A is in the direction of the tangent AC; a short time δt later at B, its velocity v_B is in the direction of the tangent BD. Since their directions are different, the velocity v_B is different from the velocity v_A, although their magnitudes are both equal to v. Thus a velocity change or acceleration has occurred from A to B.

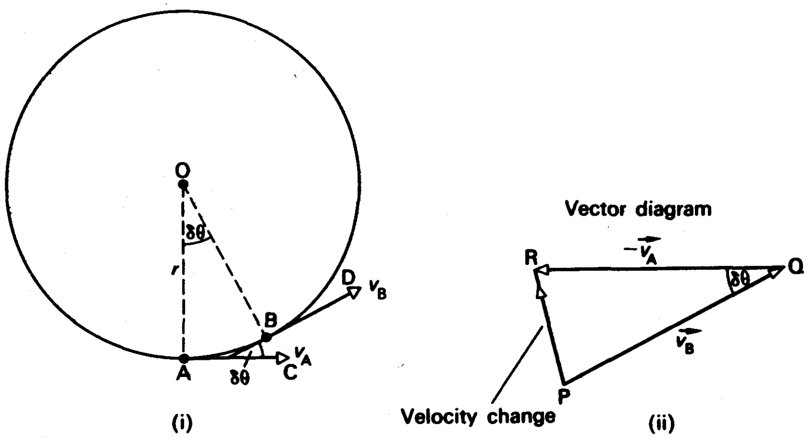

FIG. 2.2 Acceleration in circle

The velocity change from A to B $= \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$. The arrows denote vector quantities. In Fig. 2.2 (ii), PQ is drawn to represent v_B in magnitude (v) and direction (BD); QR is drawn to represent $(-\vec{v}_A)$ in magnitude (v) and direction (CA). Then, as shown on p. 11,

$$\text{velocity change} = \vec{v}_B + (-\vec{v}_A) = \vec{PR}.$$

When δt is small, the angle AOB or $\delta\theta$ is small. Thus angle PQR, equal to $\delta\theta$, is small. PR then points towards O, the centre of the circle. *The velocity change or acceleration is thus directed towards the centre.*

The magnitude of the acceleration, a, is given by

$$a = \frac{\text{velocity change}}{\text{time}} = \frac{\text{PR}}{\delta t}.$$

$$= \frac{v\,.\,\delta\theta}{\delta t}.$$

since PR $= v\,.\,\delta\theta$. In the limit, when δt approaches zero, $\delta\theta/\delta t = d\theta/dt = \omega$, the angular velocity. But $v = r\omega$ (p. 36). Hence, since $a = v\omega$,

$$a = \frac{v^2}{r} \quad \text{or} \quad r\omega^2.$$

Thus an object moving in a circle of radius r with a constant speed v has a constant acceleration towards the centre equal to v^2/r or $r\omega^2$.

Centripetal forces

The force F required to keep an object of mass m moving in a circle of radius $r = ma = mv^2/r$. It is called a *centripetal force* and acts *towards the centre* of the circle. When a stone A is whirled in a horizontal circle of centre O by means of a string, the tension T provides the centripetal force. Fig. 2.3 (i). For a racing car moving round a circular track, the friction at the wheels provides the centripetal force. Planets such as P, moving in a circular orbit round the sun S, have a centripetal force due to gravitational attraction between S and P (p. 59). Fig. 2.3 (ii).

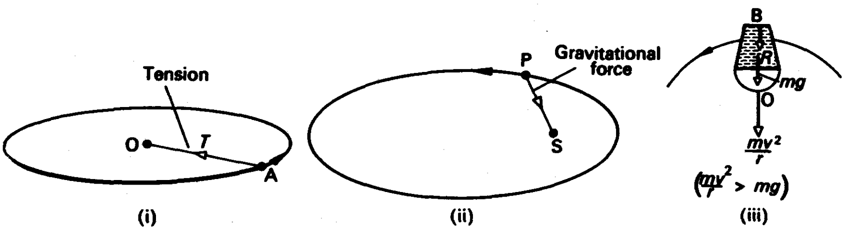

FIG. 2.3 Centripetal forces

If some water is placed in a bucket B attached to the end of a string, the bucket can be whirled in a vertical plane without any water falling out. When the bucket is vertically above the point of support O, the weight mg of the water is less than the required force mv^2/r towards the centre and so the water stays in. Fig. 2.3 (iii). The reaction R of the bucket base on the water provides the rest of the force. If the bucket is whirled slowly and $mg > mv^2/r$, part of the weight provides the force mv^2/r. The rest of the weight causes the water to accelerate downward and hence to leave the bucket.

Centrifuges

Centrifuges are used to separate particles in suspension from the less dense liquid in which they are contained. This mixture is poured into a tube in the centrifuge, which is then whirled at high speed in a horizontal circle.

The pressure gradient due to the surrounding liquid at a particular distance, r say, from the centre provides a centripetal force of $mr\omega^2$ for a small volume of liquid of mass m, where ω is the angular velocity.

If the volume of liquid is replaced by an equal volume of particles of smaller mass m' than the liquid, the centripetal force acting on the particles at the same place is then greater than that required by $(m-m')r\omega^2$. The net force urges the particles towards the centre in spiral paths, and here they collect. Thus when the centrifuge is stopped, and the container or tube assumes a vertical position, the suspension is found at the top of the tube and clear liquid at the bottom. For the same reason, cream is separated from the denser milk by spinning the mixture in a vessel. The cream spirals towards the centre and collects here.

Motion of Bicycle Rider Round Circular Track

When a person on a bicycle rides round a circular racing track, the frictional force F at the ground provides the inward force towards the centre or centripetal force. Fig. 2.4. This produces a moment about his centre of gravity G which is counterbalanced, when he leans inwards, by the moment of the normal reaction R. Thus provided no skidding occurs, $F.h = R.a = mg.a$, since $R = mg$ for no vertical motion.

$$\therefore \frac{a}{h} = \tan\theta = \frac{F}{mg},$$

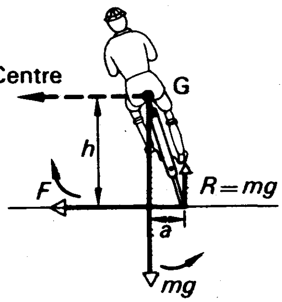

FIG. 2.4 Rider on circular track

where θ is the angle of inclination to the vertical. Now $F = mv^2/r$.

$$\therefore \tan\theta = \frac{v^2}{rg}.$$

When F is greater than the limiting friction, skidding occurs. In this case $F > \mu mg$, or $mg\tan\theta > \mu mg$. Thus $\tan\theta > \mu$ is the condition for skidding.

Motion of Car (or Train) Round Circular Track

Suppose a car (or train) is moving with a velocity v round a horizontal circular track of radius r, and let R_1, R_2 be the respective normal re-

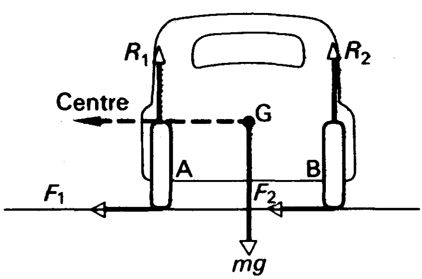

FIG. 2.5 Car on circular track

actions at the wheels A, B, and F_1, F_2 the corresponding frictional forces, Fig. 2.5. Then, for circular motion we have

$$F_1+F_2 = \frac{mv^2}{r}, \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and vertically $$R_1+R_2 = mg. \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Also, taking moments about G,

$$(F_1+F_2)h+R_1a-R_2a = O \quad . \quad . \quad . \quad \text{(iii)}$$

where $2a$ is the distance between the wheels, assuming G is mid-way between the wheels, and h is the height of G above the ground. From these three equations, we find

$$R_2 = \tfrac{1}{2}m\left(g+\frac{v^2h}{ra}\right)$$

and, vertically, $$R_1 = \tfrac{1}{2}m\left(g-\frac{v^2h}{ra}\right).$$

R_2 never vanishes since it always has a positive value. But if $v^2 = arg/h$, $R_1 = 0$, and the car is about to overturn outwards. R_1 will be positive if $v^2 < arg/h$.

Motion of Car (or Train) Round Banked Track

Suppose a car (or train) is moving round a banked track in a circular path of horizontal radius r, Fig. 2.6. If the only forces at the wheels

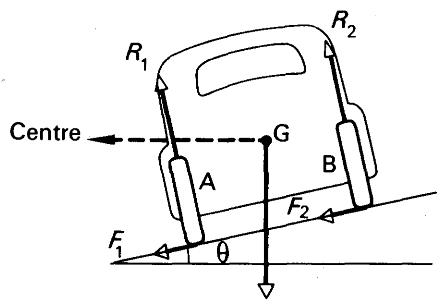

FIG. 2.6 Car on banked track

A, B are the normal reactions R_1, R_2 respectively, that is, there is no side-slip or strain at the wheels, the force towards the centre of the track is $(R_1+R_2)\sin\theta$, where θ is the angle of inclination of the plane to the horizontal.

$$\therefore (R_1+R_2)\sin\theta = \frac{mv^2}{r} \quad . \quad . \quad . \quad \text{(i)}$$

For vertical equilibrium, $(R_1+R_2)\cos\theta = mg$ . . . (ii)

Dividing (i) by (ii), $$\therefore \tan\theta = \frac{v^2}{rg} \quad . \quad . \quad . \quad \text{(iii)}$$

Thus for a given velocity v and radius r, the angle of inclination of the track for no side-slip must be $\tan^{-1}(v^2/rg)$. As the speed v increases, the angle θ increases, from (iii). A racing-track is made saucer-shaped because at higher speeds the cars can move towards a part of the track which is steeper and sufficient to prevent side-slip. The outer rail of a curved railway track is raised about the inner rail so that the force towards the centre is largely provided by the component of the reaction at the wheels. It is desirable to bank a road at corners for the same reason as a racing track is banked.

Thrust at Ground

Suppose now that the car (or train) is moving at such a speed that the frictional forces at A, B are F_1, F_2 respectively, each acting towards the centre of the track. Resolving horizontally,

$$\therefore (R_1+R_2)\sin\theta+(F_1+F_2)\cos\theta = \frac{mv^2}{r} \quad . \quad . \quad \text{(i)}$$

Resolving vertically,

$$\therefore (R_1+R_2)\cos\theta-(F_1+F_2)\sin\theta = mg \quad . \quad . \quad . \quad \text{(ii)}$$

Solving, we find

$$F_1+F_2 = m\left(\frac{v^2}{r}\cos\theta-g\sin\theta\right) \quad . \quad . \quad . \quad \text{(iii)}$$

If $\frac{v^2}{r}\cos\theta > g\sin\theta$, then (F_1+F_2) is positive; and in this case both the thrusts on the wheels at the ground are towards the centre of the track.

If $\frac{v^2}{r}\cos\theta < g\sin\theta$, then (F_1+F_2) is negative. In this case the forces F_1 and F_2 act outwards away from the centre of the track.

For stability, we have, by moments about G,

$$(F_1+F_2)h+R_1a-R_2a = 0$$

$$\therefore (F_1+F_2)\frac{h}{a} = R_2-R_1.$$

From (iii),

$$\therefore \frac{mh}{a}\left(\frac{v^2}{r}\cos\theta-g\sin\theta\right) = R_2-R_1 \quad . \quad . \quad . \quad \text{(iv)}$$

The reactions R_1, R_2 can be calculated by finding (R_1+R_2) from equations (i), (ii), and combining the result with equation (iv). This is left as an exercise to the student.

Variation of g with latitude

The acceleration due to gravity, g, varies over the earth's surface. This is due to two main causes. Firstly, the earth is elliptical, with the polar radius, b, $6{\cdot}357\times10^6$ metre and the equatorial radius, a, $6{\cdot}378\times10^6$ metre, and hence g is greater at the poles than at the equator, where the body is further away from the centre of the earth. Secondly, the earth rotates about the polar axis, AB. Fig. 2.7. We shall consider this effect in more detail, and suppose the earth is a perfect sphere.

In general, an object of mass m suspended by a spring-balance at a point on the earth would be acted on by an upward force $T = mg'$, where g' is the observed or apparent acceleration due to gravity. There would also be a downward attractive force mg towards the centre of the earth, where g is the acceleration in the absence of rotation.

(1) *At the poles*, A or B, there is no rotation. Hence $mg - T = 0$, or $mg = T = mg'$. Thus $g' = g$.

(2) *At the equator*, C or D, there is a resultant force $mr\omega^2$ towards the centre where r is the earth's radius. Since OD is the vertical, we have

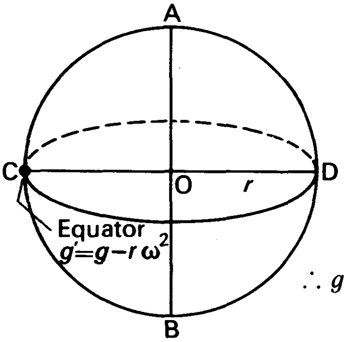

FIG. 2.7 Variation of g

$$mg - T = mr\omega^2.$$

$$\therefore T = mg - mr\omega^2 = mg'$$

$$\therefore g' = g - r\omega^2.$$

The radius r of the earth is about $6{\cdot}37 \times 10^6$ m, and $\omega = [(2\pi/(24 \times 3600)]$ radian per second.

$$\therefore g - g' = r\omega^2 = \frac{6{\cdot}37 \times 10^6 \times (2\pi)^2}{(24 \times 3600)^2} = 0{\cdot}034.$$

Latest figures give g, at the pole, $9{\cdot}832$ m s^{-2}, and g', at the equator, $9{\cdot}780$ m s^{-2}, a difference of $0{\cdot}052$ m s^{-2}. The earth's rotation accounts for $0{\cdot}034$ m s^{-2}.

EXAMPLE

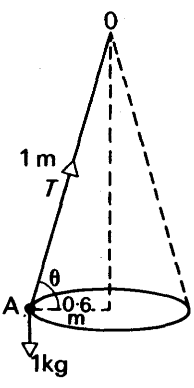

FIG. 2.8 Example

Explain the action of a centrifuge when used to hasten the deposition of a sediment from a liquid.

A pendulum bob of mass 1 kg is attached to a string 1 m long and made to revolve in a horizontal circle of radius 60 cm. Find the period of the motion and the tension of the string. (*C.*)

First part. See text, p. 38.

Second part. Suppose A is the bob, and OA is the string, Fig. 2.8. If T is the tension in newton, and θ is the angle of inclination of OA to the horizontal, then, for motion in the circle of radius $r = 60$ cm $= 0{\cdot}6$ m,

$$T\cos\theta = \frac{mv^2}{r} = \frac{mv^2}{0{\cdot}6} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Since the bob A does not move in a vertical direction, then

$$T\sin\theta = mg \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Now $\cos\theta = \frac{60}{100} = \frac{3}{5}$; hence $\sin\theta = \frac{4}{5}$.

From (ii),

$$\therefore T = \frac{mg}{\sin\theta} = \frac{1 \times 9{\cdot}8}{4/5} = 12{\cdot}25 \text{ newton.}$$

From (i) $$v = \sqrt{\frac{0{\cdot}6T\cos\theta}{m}}$$

$$= \sqrt{\frac{0{\cdot}6\times 12{\cdot}25\times 3}{1\times 5}} = 2{\cdot}1\ \text{m s}^{-1}$$

$$\therefore \text{ angular velocity, } \omega = \frac{v}{r} = \frac{2{\cdot}1}{0{\cdot}6} = \frac{7}{2}\ \text{rad s}^{-1}$$

$$\therefore \text{ period, } T, = \frac{2\pi}{\omega} = \frac{2\pi}{7/2} = \frac{4\pi}{7}\ \text{second.}$$

$$\therefore T = 1{\cdot}8\ \text{second.}$$

SIMPLE HARMONIC MOTION

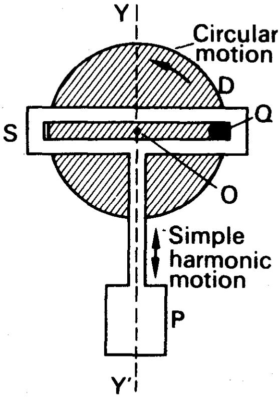

FIG. 2.9
Simple harmonic motion

When the bob of a pendulum moves to-and-fro through a small angle, the bob is said to be moving with *simple harmonic motion*. The prongs of a sounding tuning fork, and the layers of air near it, are moving with simple harmonic motion, and light waves can be considered due to simple harmonic variations.

Simple harmonic motion is closely associated with circular motion. An example is shown in Fig. 2.9. This illustrates an arrangement used to convert the circular motion of a disc D into the to-and-fro or simple harmonic motion of a piston P. The disc is driven about its axle O by a peg Q fixed near its rim. The vertical motion drives P up and down. Any horizontal component of the motion merely causes Q to move along the slot S. Thus the simple harmonic motion of P is the *projection* on the vertical line YY′ of the circular motion of Q.

An everyday example of an opposite conversion of motion occurs in car engines. Here the to-and-fro or 'reciprocating' motion of the piston engine is changed to a regular circular motion by connecting rods and shafts so that the wheels are turned.

Formulae in Simple Harmonic Motion

Consider an object moving round a circle of radius r and centre Z with a uniform angular velocity ω, Fig. 2.10. If CZF is a fixed diameter, the *foot* of the perpendicular from the moving object to this diameter moves from Z to C, back to Z and across to F, and then returns to Z, while the object moves once round the circle from O in an anti-clockwise direction. The to-and-fro motion along CZF of the foot of the perpendicular is defined as *simple harmonic motion*.

Suppose the object moving round the circle is at A at some instant, where angle OZA $= \theta$, and suppose the foot of the perpendicular from A to CZ is M. The acceleration of the object at A is $\omega^2 r$, and this

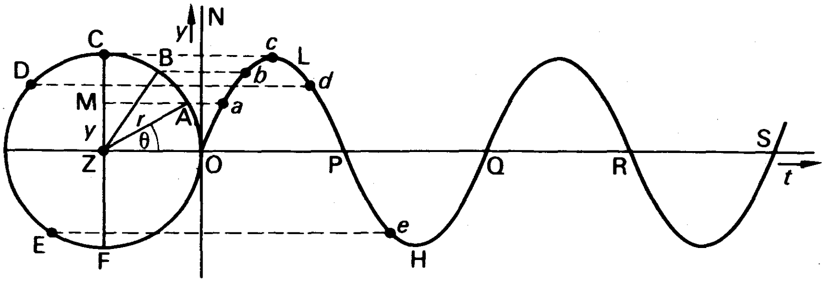

FIG. 2.10 Simple harmonic curve

acceleration is directed along the radius AZ (see p. 37). Hence the acceleration of M towards Z

$$= \omega^2 r \cos \text{AZC} = \omega^2 r \sin \theta.$$

But $r \sin \theta = \text{MZ} = y$ say.

$$\therefore \text{ acceleration of M towards Z} = \omega^2 y.$$

Now ω^2 is a constant.

$\therefore$ *acceleration of M towards Z $\propto$ distance of M from Z.*

If we wish to express mathematically that the acceleration is always directed towards Z, we must say

$$\text{acceleration towards Z} = -\omega^2 y \quad . \quad . \quad (1)$$

The minus indicates, of course, that the object begins to retard as it passes the centre, Z, of its motion. If the minus were omitted from equation (1) the latter would imply that the acceleration increases as y increases, and the object would then never return to its original position.

We can now form a definition of simple harmonic motion. It is the motion of a particle *whose acceleration is always* (*i*) *directed towards a fixed point,* (ii) *directly proportional to its distance from that point.*

Period, Amplitude. Sine Curve

The time taken for the foot of the perpendicular to move from C to F and back to C is known as the *period* (T) of the simple harmonic motion. In this time, the object moving round the circle goes exactly once round the circle from C; and since ω is the angular velocity and 2π radians (360°) is the angle described, the period T is given by

$$T = \frac{2\pi}{\omega} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The distance ZC, or ZF, is the maximum distance from Z of the foot of the perpendicular, and is known as the *amplitude* of the motion. It is equal to r, the radius of the circle.

We have now to consider the variation with time, t, of the distance,

y, from Z of the foot of the perpendicular. The distance $y = \text{ZM} = r \sin \theta$. But $\theta = \omega t$, where ω is the angular velocity.

$$\therefore y = r \sin \omega t \qquad (2)$$

The graph of y v. t is shown in Fig. 2.10, where ON represents the y-axis and OS the t-axis; since the angular velocity of the object moving round the circle is constant, θ is proportional to the time t. Thus as the foot of the perpendicular along CZF moves from Z to C and back to Z, the graph OLP is traced out; as the foot moves from Z to F and returns to Z, the graph PHQ is traced out. The graph is a *sine curve*. The complete set of values of y from O to Q is known as a cycle. The number of cycles per second is called the *frequency*. The unit '1 cycle per second' is called '1 *hertz* (*Hz*)'. The mains frequency in Great Britain is 50 Hz or 50 cycles per second.

Velocity during S.H.M.

Suppose the object moving round the circle is at A at some instant, Fig. 2.10. The velocity of the object is $r\omega$, where r is the radius of the circle, and it is directed along the tangent at A. Consequently the velocity parallel to the diameter FC at this instant $= r\omega \cos \theta$, by resolving.

$$\therefore \text{velocity, } v, \text{ of M along FC} = r\omega \cos \theta.$$

But
$$y = r \sin \theta$$

$$\therefore \cos \theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - y^2/r^2} = \frac{1}{r}\sqrt{r^2 - y^2}$$

$$\therefore v = \omega\sqrt{r^2 - y^2} \qquad (1)$$

This is the expression for the velocity of an object moving with simple harmonic motion. The maximum velocity, v_m, corresponds to $y = 0$, and hence

$$v_m = \omega r. \qquad (2)$$

Summarising our results:

(1) If the acceleration a of an object $= -\omega^2 y$, where y is the distance or displacement of the object from a fixed point, the motion is simple harmonic motion.

(2) The *period*, T, of the motion $= 2\pi/\omega$, where T is the time to make a complete to-and-fro movement or cycle. The *frequency*, f, $= 1/T$ and its unit is 'Hz'.

(3) The amplitude, r, of the motion is the maximum distance on either side of the centre of oscillation.

(4) The velocity at any instant, v, $= \omega\sqrt{r^2 - y^2}$; the maximum velocity $= \omega r$. Fig. 2.11 (i) shows a graph of the variation of v and acceleration a with displacement y, which are respectively an ellipse and a straight line.

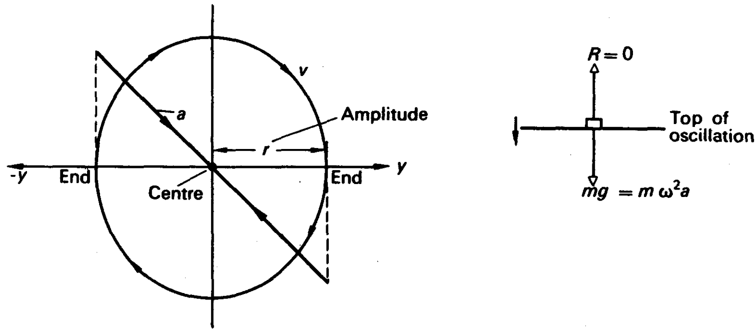

FIG. 2.11 Simple harmonic motion

S.H.M. and g

If a small coin is placed on a horizontal platform connected to a vibrator, and the amplitude is kept constant as the frequency is increased from zero, the coin will be heard 'chattering' at a particular frequency f_0. At this stage the reaction of the table with the coin becomes zero at some part of every cycle, so that it loses contact periodically with the surface. Fig. 2.11 (ii).

The maximum acceleration in S.H.M. occurs at the end of the oscillation because the acceleration is directly proportional to the displacement. Thus maximum acceleration $= \omega^2 a$, where a is the amplitude and ω is $2\pi f_0$.

The coin will lose contact with the table when it is moving *down* with acceleration g (Fig. 2.11 (ii)). Suppose the amplitude is 8·0 cm. Then

$$(2\pi f_0)^2 a = g$$

$$\therefore\ 4\pi^2 f_0^2 \times 0{\cdot}08 = 9{\cdot}8$$

$$\therefore f_0 = \sqrt{\frac{9{\cdot}8}{4\pi^2 \times 0{\cdot}08}} = 1{\cdot}8 \text{ Hz.}$$

Damping of S.H.M.

In practice, simple harmonic variations of a pendulum, for example, will die away as the energy is dissipated by viscous forces due to the air. The oscillation is then said to be *damped.* In the absence of any damping forces the oscillations are said to be *free.*

A simple experiment to investigate the effect of damping is illustrated in Fig. 2.12 (i). A suitable weight A is suspended from a helical spring S, a pointer P is attached to S, and a vertical scale R is set up behind P. The weight A is then set pulled down and released. The period, and the time taken for the oscillations to die away, are noted.

As shown in Fig. 2.12 (ii), A is now fully immersed in a damping medium, such as a light oil, water or glycerine. A is then set oscillating,

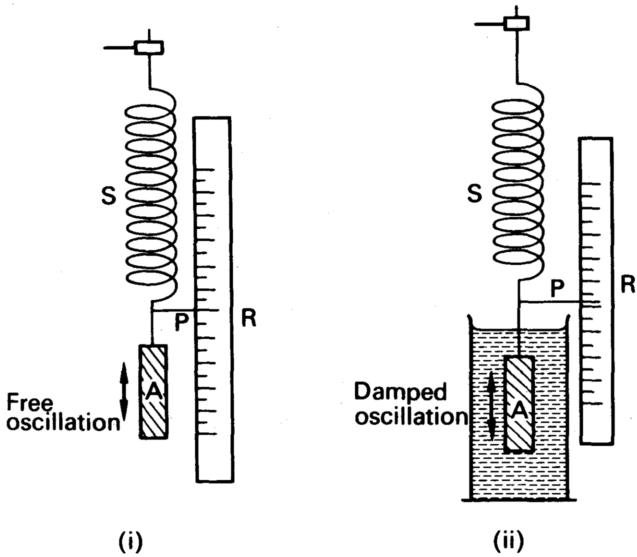

FIG. 2.12 Experiment on damped oscillations

and the time for oscillations to die away is noted. It is shorter than before and least for the case of glycerine. The decreasing amplitude in successive oscillations may also be noted from the upward limit of travel of P and the results plotted.

Fig. 2.13 (i), (ii) shows how damping produces an exponential fall in the amplitude with time.

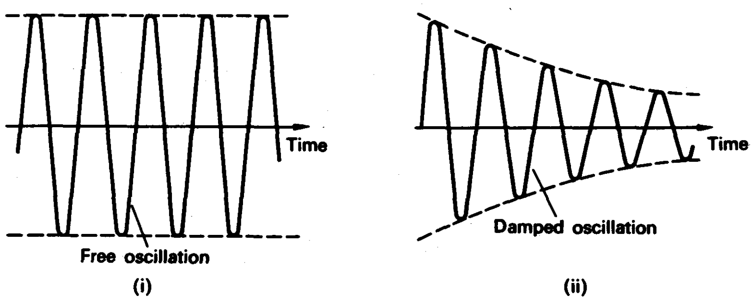

FIG. 2.13 Free and damped oscillations

The experiment works best for a period of about $\frac{1}{2}$-second and a weight which is long and thin so that the damping is produced by non-turbulent fluid flow over the vertical sides. During the whole cycle, A must be totally immersed in the fluid.

EXAMPLE

A steel strip, clamped at one end, vibrates with a frequency of 20 Hz and an amplitude of 5 mm at the free end, where a small mass of 2 g is positioned.

Find (*a*) the velocity of the end when passing through the zero position, (*b*) the acceleration at maximum displacement, (*c*) the maximum kinetic and potential energy of the mass.

Suppose $y = r \sin \omega t$ represents the vibration of the strip where r is the amplitude.

(*a*) The velocity, $v, = \omega\sqrt{r^2 - y^2}$ (p. 45). When the end of the strip passes through the zero position $y = 0$; and the maximum speed, v_m, is given by

$$v_m = \omega r.$$

Now $\omega = 2\pi f = 2\pi \times 20$, and $r = 0{\cdot}005$ m.

$$\therefore\ v_m = 2\pi \times 20 \times 0{\cdot}005 = 0{\cdot}628 \text{ m s}^{-1}.$$

(*b*) The acceleration $= -\omega^2 y = -\omega^2 r$ at the maximum displacement.

$$\therefore \text{ acceleration} = (2\pi \times 20)^2 \times 0{\cdot}005$$
$$= 79 \text{ m s}^{-2}.$$

(c) $m = 2 \text{ g} = 2 \times 10^{-3}$ kg, $v_m = 0{\cdot}628 \text{ m s}^{-1}$.

$\therefore$ maximum K.E. $= \frac{1}{2}mv_m^{\ 2} = \frac{1}{2} \times (2 \times 10^{-3}) \times 0{\cdot}628^2 = 3{\cdot}9 \times 10^{-4}$ J (approx.).

Maximum P.E. ($v = 0$) = Maximum K.E. $= 3{\cdot}9 \times 10^{-4}$ J.

Simple Pendulum

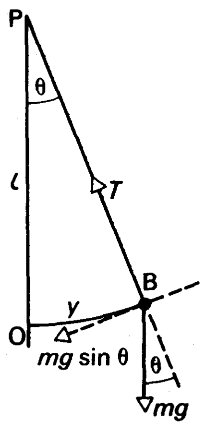

FIG. 2.14
Simple pendulum

We shall now study some cases of simple harmonic motion. Consider a *simple pendulum*, which consists of a small mass m attached to the end of a length l of wire, Fig. 2.14. If the other end of the wire is attached to a fixed point P and the mass is displaced slightly, it oscillates to-and-fro along the arc of a circle of centre P. We shall now show that the motion of the mass about its original position O is simple harmonic motion.

Suppose that the vibrating mass is at B at some instant, where OB $= y$ and angle OPB $= \theta$. At B, the force pulling the mass towards O is directed along the tangent at B, and is equal to $mg \sin \theta$. The tension, T, in the wire has no component in this direction, since PB is perpendicular to the tangent at B. Thus, since force = mass × acceleration (p. 13),

$$-mg \sin \theta = ma,$$

where a is the acceleration along the arc OB; the minus indicates that the force is towards O, while the displacement, y, is measured along the arc from O in the opposite direction. *When θ is small*, $\sin \theta = \theta$ in radians; also $\theta = y/l$. Hence,

$$-mg\theta = -mg\frac{y}{l} = ma$$

$$\therefore\ a = -\frac{g}{l}y = -\omega^2 y,$$

where $\omega^2 = g/l$. Since the acceleration is proportional to the distance y from a fixed point, the motion of the vibrating mass is simple harmonic motion (p. 50). Further, from p. 50, the period $T = 2\pi/\omega$.

$$\therefore \mathbf{T} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{\mathbf{l}}{\mathbf{g}}} \quad . \quad . \quad . \quad (1)$$

At a given place on the earth, where g is constant, the formula shows that the period T depends only on the length, l, of the pendulum. Moreover, the period remains constant even when the amplitude of the vibrations diminish owing to the resistance of the air. This result was first obtained by Galileo, who noticed a swinging lantern one day, and timed the oscillations by his pulse. He found that the period remained constant although the swings gradually diminished in amplitude.

Determination of g by Simple Pendulum

The acceleration due to gravity, g, can be found by measuring the period, T, of a simple pendulum corresponding to a few different lengths, l, from 80 cm to 180 cm for example. To perform the experiment accurately: (i) Fifty oscillations should be timed, (ii) a small angle of swing is essential, less than 10°, (iii) a small sphere should be tied to the end of a thread to act as the mass, and its radius added to the length of the thread to determine l.

A graph of l against T^2 is now plotted from the results, and a straight line AB, which should pass through the origin, is then drawn to lie evenly between the points, Fig. 2.15.

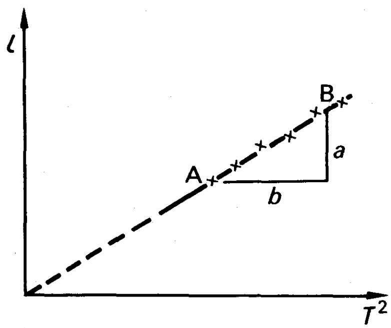

FIG. 2.15 Graph of l v. T^2

Now

$$T = 2\pi\sqrt{\frac{l}{g}},$$

$$\therefore T^2 = \frac{4\pi^2 l}{g}$$

$$\therefore g = 4\pi^2 \times \frac{l}{T^2} \quad . \quad . \quad . \quad . \quad (1)$$

The gradient a/b of the line AB is the magnitude of l/T^2; and by substituting in (1), g can then be calculated.

If the pendulum is suspended from the ceiling of a very tall room and the string and bob reaches nearly to the floor, then one may proceed to find g by (i) measuring the period T_1, (ii) cutting off a measured length a of the string and determining the new period T_2 with the shortened string. Then, if h is the height of the ceiling above the bob initially, $T_1 = 2\pi\sqrt{h/g}$ and $T_2 = 2\pi\sqrt{(h-a)/g}$. Thus

$$h = \frac{gT_1^2}{4\pi^2} \quad \text{and} \quad h-a = \frac{gT_2^2}{4\pi^2}.$$

$$\therefore a = \frac{g}{4\pi^2}(T_1^2 - T_2^2).$$

$$\therefore g = \frac{4\pi^2 a}{T_1^2 - T_2^2}.$$

Thus g can be calculated from a, T_1 and T_2. Alternatively, the period T can be measured for several lengths a. Then, since $T = 2\pi\sqrt{(h-a/g}$,

$$h-a = \frac{g}{4\pi^2}T^2.$$

A graph of a v. T^2 is thus a straight line whose gradient is $g/4\pi^2$. Hence g can be found. The intercept on the axis of a, when $T^2 = 0$, is h, the height of the ceiling above the bob initially.

The Spiral Spring or Elastic Thread

When a weight is suspended from the end of a spring or an elastic thread, experiment shows that the extension of the spring, i.e., the increase in length, is proportional to the weight, provided that the elastic limit of the spring is not exceeded (see p. 181). Generally, then, *the tension (force), T, in a spring is proportional to the extension x produced*, i.e., $T = kx$, where k is a constant of the spring.

Consider a spring or an elastic thread PA of length l suspended from a fixed point P, Fig. 2.16. When a mass m is placed on it, the spring stretches to O by a length e given by

$$mg = ke, \quad . \quad . \quad . \quad \text{(i)}$$

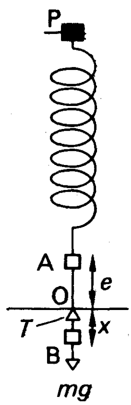

FIG. 2.16 Spiral spring

since the tension in the spring is then mg. If the mass is pulled down a little and then released, it vibrates up-and-down above and below O. Suppose at an instant that B is at a distance x below O. The tension T of the spring at B is then equal to $k(e+x)$, and hence the force towards O $=$ $k(e+x)-mg$. Since force $=$ mass $\times$ acceleration,

$$\therefore -[k(a+x)-mg] = ma,$$

the minus indicates that the net force is upward at this instant, whereas the displacement x is measured from O in the opposite direction at the same instant. From this equation,

$$-ke-kx+mg = ma.$$

But, from (i),

$$mg = ke,$$

$$\therefore -kx = ma,$$

$$\therefore a = -\frac{k}{m}x = -\omega^2 x,$$

where $\omega^2 = k/m$. Thus the motion is simple harmonic about O, and the period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, since $mg = ke$, it follows that $m/k = e/g$.

$$\therefore T = 2\pi\sqrt{\frac{e}{g}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1), it follows that $T^2 = 4\pi^2 m/k$. Consequently a graph of T^2 v. m should be a straight line passing through the origin. In practice, when the load m is varied and the corresponding period T is measured, a straight line graph is obtained when T^2 is plotted against m, thus verifying indirectly that the motion of the load was simple harmonic. The graph does not pass through the origin, however, owing to the mass and the movement of the various parts of the spring. This has not been taken into account in the foregoing theory and we shall now show how g may be found in this case.

Determination of g by Spiral Spring

The mass s of a vibrating spring is taken into account, in addition to the mass m suspended at the end, theory beyond the scope of this book then shows that the period of vibration, T, is given by

$$T = 2\pi\sqrt{\frac{m+\lambda s}{k}} \quad . \quad . \quad . \quad . \quad . \quad (i)$$

where λ is approximately $\frac{1}{3}$ and k is the elastic constant of the spring. Squaring (i) and re-arranging,

$$\therefore \frac{k}{4\pi^2}T^2 = m+\lambda s \quad . \quad . \quad . \quad . \quad . \quad (ii)$$

Thus, since λ, k, s are constants, a graph of T^2 v. m should be a straight line when m is varied and T observed. A straight line graph verifies indirectly that the motion of the mass at the end of the spring is simple harmonic. Further, the magnitude of $k/4\pi^2$ can be found from the slope of the line, and hence k can be calculated.

If a mass M is placed on the end of the spring, producing a steady extension e less than the elastic limit, then $Mg = ke$.

$$\therefore g = \frac{e}{M} \times k \quad . \quad . \quad . \quad . \quad . \quad . \quad (iii)$$

By attaching different masses to the spring, and measuring the corresponding extension, the magnitude of e/M can be found by plotting e v. M and measuring the slope of the line. This is called the 'static' experiment on the spring. From the magnitude of k obtained in the 'dynamic' experiment when the period was determined for different loads, the value of g can be found by substituting the magnitudes of e/M and k in (iii).

Oscillations of a Liquid in a U-Tube

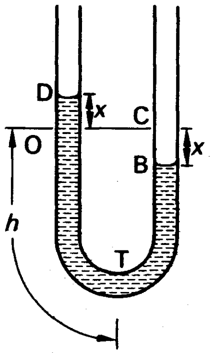

FIG. 2.17
S.H.M. of liquid

If the liquid on one side of a U-tube T is depressed by blowing gently down that side, the levels of the liquid will oscillate for a short time about their respective initial positions O, C, before finally coming to rest, Fig. 2.17.

The period of oscillation can be found by supposing that the level of the liquid on the left side of T is at D at some instant, at a height x above its original (undisturbed) position O. The level B of the liquid on the other side is then at a depth x below its original position C, and hence the excess pressure on the whole liquid, as shown on p. 110,

$$= \text{excess height} \times \text{density of liquid} \times g$$

$$= 2x\rho g.$$

Now pressure = force per unit area.

$\therefore$ force on liquid = pressure × area of cross-section of the tube

$$= 2x\rho g \times A,$$

where A is the cross-sectional area of the tube.

This force causes the liquid to accelerate. The mass of liquid in the U-tube = volume × density = $2hA\rho$, where $2h$ is the total length of the liquid in T. Now the acceleration, a, towards O or C is given by *force* = *mass* × *a*.

$$\therefore \quad -2x\rho g A = 2hA\rho a.$$

The minus indicates that the force towards O is opposite to the displacement measured from O at that instant.

$$\therefore \ a = -\frac{g}{h}x = -\omega^2 x,$$

where $\omega^2 = \frac{g}{h}$. The motion of the liquid about O (or C) is thus simple harmonic, and the period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}.$$

P.E. and K.E. exchanges in oscillating systems

We can now make a general point about *oscillations* and *oscillating systems.* As an illustration, suppose that one end of a spring S of negligible mass is attached to a smooth object A, and that S and A are laid on a horizontal smooth table. If the free end of S is attached to the table and A is pulled slightly to extend the spring and then released, the system vibrates with simple harmonic motion. This is the case discussed on p. 50, without taking gravity into account. The centre of oscillation O is the position of the end of the spring corresponding to its natural length, that is, when the spring is neither extended or compressed. If the spring extension obeys the law *force* = kx, where k is a constant, and m is the mass of A, then, as on p. 51, it can easily be shown that the period T of oscillation is given by:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

The energy of the stretched spring is *potential energy,* P.E.—its molecules are continually displaced or compressed relative to their normal distance apart. The P.E. for an extension $x = \int F.dx = \int kx.dx = \frac{1}{2}kx^2$.

The energy of the mass is *kinetic energy,* K.E., or $\frac{1}{2}mv^2$, where v is the velocity. Now from $x = a\sin\omega t$, $v = dx/dt = \omega a\cos\omega t$.

$$\therefore \text{ total energy of spring plus mass} = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}ka^2\sin^2\omega t + \tfrac{1}{2}m\omega^2a^2\cos^2\omega t.$$

But $\omega^2 = k/m$, or $k = m\omega^2$.

$$\therefore \text{ total energy} = \tfrac{1}{2}m\omega^2a^2(\sin^2\omega t + \cos^2\omega t) = \tfrac{1}{2}m\omega^2a^2 = \textit{constant}.$$

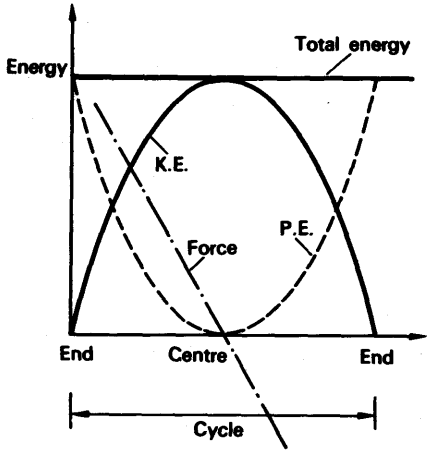

FIG. 2.18 Energy of S.H.M.

Thus the total energy of the vibrating mass and spring is constant. When the K.E. of the mass is a maximum (energy = $\frac{1}{2}m\omega^2a^2$ and mass passing through the centre of oscillation), the P.E. of the spring is then zero ($x = 0$). Conversely, when the P.E. of the spring is a maximum (energy = $\frac{1}{2}ka^2 = \frac{1}{2}m\omega^2a^2$ and mass at end of the oscillation), the K.E. of the mass is zero ($v = 0$). Fig. 2.18 shows the variation of P.E. and K.E. with displacement x; the force F extending the spring, also shown, is directly proportional to the displacement from the centre of oscillation.

The constant interchange of energy between potential and kinetic energies is essential for producing and maintaining oscillations, whatever their nature. In the case of the oscillating bob of a simple pendulum, for example, the bob loses kinetic energy after passing through the middle of the swing, and then stores the energy as potential energy as it rises to the top of the swing. The reverse occurs as it swings back. In the case of oscillating layers of air when a sound wave passes, kinetic energy of the moving air molecules is converted to potential energy when the air is compressed. In the case of electrical oscillations, a coil L and a capacitor C in the circuit constantly exchange energy; this is stored alternately in the magnetic field of L and the electric field of C.

EXAMPLES

1. Define *simple harmonic motion* and state the relation between displacement from its mean position and the restoring force when a body executes simple harmonic motion.

A body is supported by a spiral spring and causes a stretch of 1·5 cm in the spring. If the mass is now set in vertical oscillation of small amplitude, what is the periodic time of the oscillation? (*L.*)

First part. Simple harmonic motion is the motion of an object whose acceleration is proportional to its distance from a fixed point and is always directed towards that point. The relation is: Restoring force $= -k \times$ distance from fixed point, where k is a constant.

Second part. Let m be the mass of the body in kg. Then, since 1·5 cm = 0·015 m

$$mg = k \times 0{\cdot}015 \qquad \text{(i)}$$

where k is a constant of the spring in N m^{-1}. Suppose the vibrating body is x m below its original position at some instant and is moving downwards. Then since the extension is $(x+0{\cdot}015)$ m, the net downward force

$$= mg - k(x+0{\cdot}015)$$

$$= mg - k \times 0{\cdot}015 - kx = -kx$$

from (i). Now mass × acceleration = force.

$$\therefore\ m \times \text{acceleration} = -kx$$

$$\therefore\ \text{acceleration} = -\frac{k}{m}x.$$

But, from (i),

$$\frac{k}{m} = \frac{g}{0{\cdot}015}$$

$$\therefore\ \text{acceleration} = \frac{g}{0{\cdot}015}x = -\omega^2 x,$$

where $\omega^2 = g/0{\cdot}015$.

$$\therefore\ \text{period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0{\cdot}015}{g}} = 2\pi\sqrt{\frac{0{\cdot}015}{9{\cdot}8}}$$

$$= 0{\cdot}25 \text{ second.}$$

2. A small bob of mass 20 g oscillates as a simple pendulum, with amplitude 5 cm and period 2 seconds. Find the velocity of the bob and the tension in the supporting thread, when the velocity of the bob is a maximum.

First part. See text.

Second part. The velocity, v, of the bob is a maximum when it passes through its original position. With the usual notation (see p. 45), the maximum velocity v_m is given by

$$v_m = \omega r,$$

where r is the amplitude of 0·05 m. Since $T = 2\pi/\omega$,

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$\therefore v_m = \omega a = \pi \times 0{\cdot}05 = 0{\cdot}16 \text{ m s}^{-1}.$$

Suppose P is the tension in the thread. The net force towards the centre of the circle along which the bob moves is then given by $(P-mg)$. The acceleration towards the centre of the circle, which is the point of suspension, is $v_m{}^2/l$, where l is the length of the pendulum.

$$\therefore P - mg = \frac{mv_m{}^2}{l}$$

$$\therefore P = mg + \frac{mv_m{}^2}{l}$$

Now
$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\therefore l = \frac{gT^2}{4\pi^2} = \frac{g\times 4}{4\pi^2} = \frac{g}{\pi^2}$$

Since $m = 0{\cdot}02$ kg, $g = 9{\cdot}8$ m s^{-2}, it follows from above that

$$P = 0{\cdot}02 \times 9{\cdot}8 + \frac{0{\cdot}02 \times (0{\cdot}05\pi)^2 \times \pi^2}{9{\cdot}8}$$

$$= 19{\cdot}65 \times 10^{-2} \text{ newton}$$

Waves. Wave equation

Waves and their properties can be demonstrated by producing them on the surface of water, as in a ripple tank. As the wave travels outwards from the centre of disturbance, it reaches more distant particles of water at a later time. Thus the particles of water vibrate out of *phase* with each other while the wave travels. It should be noted that the vibrating particles are the origin of the wave. Their mean position remains the same as the wave travels, but like the simple harmonic oscillators previously discussed, they store and release *energy* which is handed on from one part of the medium to another. The wave shows the energy travelling through the medium.

If the displacement y of a vibrating particle P is represented by $y = a \sin \omega t$, the displacement of a neighbouring particle Q can be represented by $y = a \sin(\omega t + \phi)$. ϕ is called the *phase angle* between the two vibrations. If $\phi = \pi/2$ or 90°, the vibration of Q is $y = a \sin(\omega t + \pi/2)$. In this case, $y = 0$ when $t = 0$ for P, but $y = a \sin \pi/2 = a$ when $t = 0$ for Q. Comparing the two simple harmonic variations, it can be seen that Q *leads* on P by a quarter of a period.

If the wave is 'frozen' at different times, the displacements of the various particles will vary according to their position or distance x

from some chosen origin such as the centre of disturbance. Now the *wavelength*, λ, of a wave is the distance between successive crests or troughs. At these points the phase difference is 2π. Consequently the phase angle for a distance x is $(x/\lambda)\times 2\pi$ or $2\pi x/\lambda$. The *wave equation*, which takes x into account as well as the time t, can thus be written as:

$$y = a\sin\left(2\pi\frac{t}{T}-2\pi\frac{x}{\lambda}\right) = a\sin 2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right) \quad . \quad . \quad (1)$$

Other forms of the wave equation may be used. The velocity v of a wave is the distance travelled by the disturbance in 1 second. If the frequency of the oscillations is f, then f waves travel outwards in 1 second. Each wave occupies a length λ. Hence $v = f\lambda$. Further, the period T is the time for 1 oscillation. Thus $f = 1/T$ and hence $v = f\lambda = \lambda/T$. Substituting for T in (1), the wave equation may also be written as:

$$y = a\sin\frac{2\pi}{\lambda}(vt-x) \quad . \quad . \quad . \quad (2)$$

The wave equation in (1) or (2) is a *progressive wave*. The energy of the wave travels outwards through the medium as time goes on.

Longitudinal and transverse waves

Waves can be classified according to the direction of their vibrations. A *longitudinal wave* is one produced by *vibrations parallel* to the direction of travel of the wave. An example is a sound wave. The layers of air are always vibrating in a direction parallel to the direction of travel of the wave. A longitudinal wave can be seen travelling in a 'Slinky' coil when one end is fixed and the other is pulled to-and-fro in the direction of the coil.

A *transverse wave* is one produced by *vibrations perpendicular* to the direction of travel of the wave. Light waves are transverse waves. The wave along a bowed string of a violin is a transverse wave.

Velocity of waves

There are various types of waves. A longitudinal wave such as a sound wave is a *mechanical wave*. The speed v with which the energy travels depends on the restoring stress after particles in the medium are strained from their original position. Thus v depends on the *modulus of elasticity* of the medium. It also depends on the inertia of the particles, of which the mass per unit volume or density ρ is a measure.

By dimensions, as well as rigorously, it can be shown that

$$v = \sqrt{\frac{\text{modulus of elasticity}}{\rho}}.$$

For a solid, the modulus is Young's modulus, E. Thus $v = \sqrt{E/\rho}$. For a liquid or gas, the modulus is the bulk modulus, k. Hence $v = \sqrt{k/\rho}$. In air, $k = \gamma p$, where γ is the ratio of the principal specific heats of air and p is the atmospheric pressure. Thus $v = \sqrt{\gamma p/\rho}$ (p. 163).

When a taut string is plucked or bowed, the velocity of the transverse wave along it is given by $v = \sqrt{T/m}$, where T is the tension and m is the mass per unit length of the string. In this case T provides the restoring force acting on the displaced particles of string and m is a measure of their inertia.

Electromagnetic waves, which are due to electric and magnetic vibrations, form an important group of waves in nature. Radio waves, infra-red, visible and ultra-violet light, X-rays, and γ-rays are all electromagnetic waves, ranging from long wavelength such as 1000 metres (radio waves) to short wavelengths such as 10^{-8} m (γ-waves). Unlike the mechanical waves, no material medium is needed to carry the waves. The speed of all electromagnetic waves in a vacuum is the same, about 3×10^8 metre per second. The speed varies with wavelength in material media and this explains why dispersion (separation of colours) of white light is produced by glass.

Stationary waves

The equation $y = a\sin 2\pi(t/T - x/\lambda)$ represents a progressive wave travelling in the x-direction. A wave of the same amplitude and frequency travelling in the *opposite direction* is represented by the same form of equation but with $-x$ in place of x, that is, by $y = a\sin 2\pi(t/T + x/\lambda)$.

The *principle of superposition* states that the combined effect or resultant of two waves in a medium can be obtained by adding the displacements at each point due to the respective waves. Thus if the displacement due to one wave is represented by y_1, and that due to the other wave by y_2, the resultant displacement y is given by

$$y = y_1 + y_2 = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

$$= 2a\sin 2\pi\frac{t}{T}\,.\cos 2\pi\frac{x}{\lambda} = A\sin 2\pi\frac{t}{T},$$

where $A = 2a\cos 2\pi x/\lambda$.

A represents the amplitude at different points in the medium. When $x = 0$, $y = A$; when $x = \lambda/4$, $A = 0$; when $x = \lambda/2$, $y = -A$; when $x = -3\lambda/4$, $y = 0$. Thus at some points called *antinodes*, A, the amplitude of vibration is a maximum. At points half-way between the antinodes called *nodes*, N, the amplitude is zero, that is, there is no vibration here. Fig. 2.19 (i). This type of wave, which stays in one place in a medium, is called a *stationary* or *standing wave*. Stationary waves may be produced which are either longitudinal or transverse.

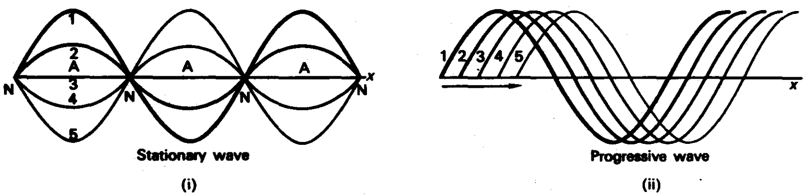

FIG. 2.19 Stationary and progressive waves

Unlike the progressive wave, where the energy travels outwards through the medium, Fig. 2.19 (ii), the energy of the stationary wave remains stored in one part of the medium. Stationary waves are produced in musical instruments when they are played. Stationary radio waves are also produced in receiving aerials. Stationary waves, due to electron motion, are believed to be present around the nucleus of atoms.

Interference. Diffraction

A stationary wave is a special case of *interference* between two waves. Another example occurs when two tuning forks of nearly equal frequency are sounded together. A periodic variation of loud sounds called 'beats' is then heard. They are due to the periodic variation of the amplitude of the resultant wave. If two very close coherent sources of light are obtained, interference between the two waves may produce bright and dark bands.

Diffraction is the name given to the interference between waves coming from coherent sources on the same undivided wavefront. The effect is pronounced when a wave is incident on a narrow opening whose width is of comparable order to the wavelength. The wave now spreads out or is 'diffracted' after passing through the slit. If the width of the slit, however, is large compared with the wavelength, the wave passes straight through the opening without any noticeable diffraction. This is why visible light, which has wavelengths of the order of 6×10^{-7} m, passes straight through wide openings and produces sharp shadows; whereas sound, which has wavelengths over a million times longer and of the order of say 0·5 m, can be heard round corners.

Further details of wave phenomena are discussed in the Sound and Optics sections of the book.

GRAVITATION

Kepler's Laws

The motion of the planets in the heavens had excited the interest of the earliest scientists, and Babylonian and Greek astronomers were able to predict their movements fairly accurately. It was considered for some time that the earth was the centre of the universe, but about 1542 COPERNICUS suggested that the planets revolved round the sun as centre. A great advance was made by KEPLER about 1609. He had studied for many years the records of observations on the planets made by TYCHO BRAHE, and he enunciated three laws known by his name. These state:

(1) The planets describe ellipses about the sun as one focus.

(2) The line joining the sun and the planet sweeps out equal areas in equal times.

(3) The squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun.

The third law was announced by Kepler in 1619.

Newton's Law of Gravitation

About 1666, at the early age of 24, NEWTON discovered a universal law known as the *law of gravitation.*

He was led to this discovery by considering the motion of a planet moving in a circle round the sun S as centre. Fig. 2.20 (i). The force acting on the planet of mass m is $mr\omega^2$, where r is the radius of the circle and ω is the angular velocity of the motion (p. 38). Since $\omega = 2\pi/T$, where T is the period of the motion,

$$\text{force on planet} = mr\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 mr}{T^2}.$$

This is equal to the force of attraction of the sun on the planet. *Assuming an inverse-square law*, then, if k is a constant,

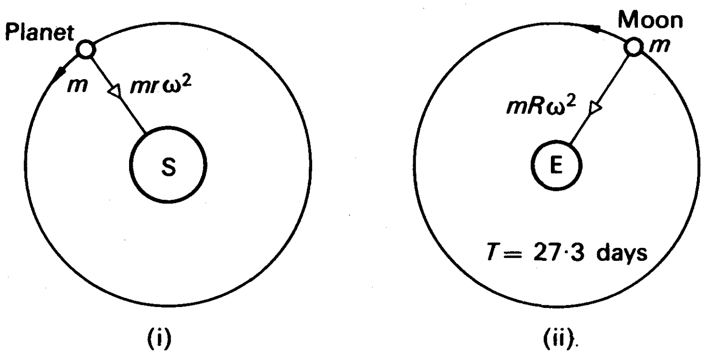

FIG. 2.20 Satellites

$$\text{force on planet} = \frac{km}{r^2}.$$

$$\therefore \frac{km}{r^2} = \frac{4\pi^2 mr}{T^2}$$

$$\therefore T^2 = \frac{4\pi^2}{k} r^3$$

$$\therefore T^2 \propto r^3,$$

since k, π are constants.

Now Kepler had announced that the squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun (see above). Newton thus suspected that *the force between the sun and the planet was inversely proportional to the square of the distance between them.* The great scientist now proceeded to test the inverse-square law by applying it to the case of the moon's motion round the earth. Fig. 2.20 (ii). The moon has a period of revolution, T, about the earth of approximately 27·3 days, and the force on it $= mR\omega^2$, where R is the radius of the moon's orbit and m is its mass.

$$\therefore \text{ force} = mR\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 mR}{T^2}.$$

If the planet were at the earth's surface, the force of attraction on it due to the earth would be mg, where g is the acceleration due to gravity. Fig. 2.20 (ii). Assuming that the force of attraction varies as the inverse square of the distance between the earth and the moon,

$$\therefore \frac{4\pi^2 mR}{T^2} : mg = \frac{1}{R^2} : \frac{1}{r^2},$$

where r is the radius of the earth.

$$\therefore \frac{4\pi^2 R}{T^2 g} = \frac{r^2}{R^2},$$

$$\therefore g = \frac{4\pi^2 R^3}{r^2 T^2} \quad . \quad . \quad . \quad . \quad (1)$$

Newton substituted the then known values of R, r, and T, but was disappointed to find that the answer for g was not near to the observed value, 9·8 m s^{-2}. Some years later, he heard of a new estimate of the radius of the moon's orbit, and on substituting its value he found that the result for g was close to 9·8 m s^{-2}. Newton saw that a universal law could be formulated for the attraction between any two particles of matter. He suggested that: *The force of attraction between two given masses is inversely proportional to the square of their distance apart.*

Gravitational Constant, G, and its Determination

From Newton's law, it follows that the force of attraction, F, between two masses m, M at a distance r apart is given by $F \propto \frac{mM}{r^2}$.

$$\therefore F = G\frac{mM}{r^2}, \quad . \quad . \quad . \quad . \quad (2)$$

where G is a universal constant known as the *gravitational constant*. This expression for F is *Newton's law of gravitation.*

From (2), it follows that G can be expressed in 'N m^2 kg^{-2}'. The dimensions of G are given by

$$[G] = \frac{MLT^{-2} \times L^2}{M^2} = M^{-1}L^3T^{-2}.$$

Thus the unit of G may also be expressed as m^3 kg^{-1} s^{-2}.

A celebrated experiment to measure G was carried out by C. V. Boys in 1895, using a method similar to one of the earliest determinations of G by Cavendish in 1798. Two identical balls, a, b, of gold,

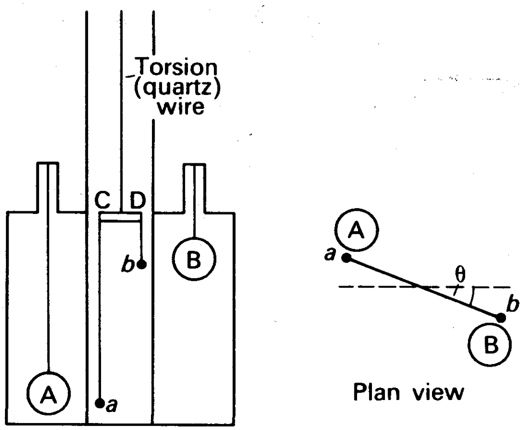

FIG. 2.21 Experiment on G

5 mm in diameter, were suspended by a long and a short fine quartz fibre respectively from the ends, C, D, of a highly-polished bar CD, Fig. 2.21. Two large identical lead spheres, A, B, 115 mm in diameter, were brought into position near a, b respectively. As a result of the attraction between the masses, two equal but opposite forces acted on CD. The bar was thus deflected, and the angle of deflection, θ, was measured by a lamp and scale method by light reflected from CD. The high sensitivity of the quartz fibres enabled the small deflection to be measured accurately, and the small size of the apparatus allowed it to be screened considerably from air convection currents.

Calculation for G

Suppose d is the distance between a, A, or b, B, when the deflection is θ. Then if m, M are the respective masses of a, A,

$$\text{torque of couple on CD} = G\frac{mM}{d^2} \times \text{CD}.$$

But $$\text{torque of couple} = c\theta,$$

where c is the torque in the torsion wire per unit radian of twist (p. 192).

$$\therefore G\frac{mM}{d^2} \times \text{CD} = c\theta.$$

$$\therefore G = \frac{c\theta d^2}{mM \times \text{CD}} \quad . \quad . \quad . \quad (1)$$

The constant c was determined by allowing CD to oscillate through a small angle and then observing its period of oscillation, T, which was of the order of 3 minutes. If I is the known moment of inertia of the system about the torsion wire, then (see p. 75),

$$T = 2\pi\sqrt{\frac{I}{c}}.$$

The constant c can now be calculated, and by substitution in (i), G can be determined. Accurate experiments showed that $G = 6{\cdot}66 \times 10^{-11}$ N m^2 kg^{-2} and Heyl, in 1942, found G to be $6{\cdot}67 \times 10^{-11}$ N m^2 kg^{-2}.

Mass and Density of Earth

At the earth's surface the force of attraction on a mass m is mg, where g is the acceleration due to gravity. Now it can be shown that it is legitimate in calculations to assume that the mass, M, of the earth is concentrated at its centre, if it is a sphere. Assuming that the earth is spherical and of radius r, it then follows that the force of attraction of the earth on the mass m is GmM/r^2.

$$\therefore\ G\frac{mM}{r^2} = mg.$$

$$\therefore\ g = \frac{GM}{r^2}.$$

$$\therefore\ M = \frac{gr^2}{G}.$$

Now, $g = 9{\cdot}8$ m s^{-2}, $r = 6{\cdot}4 \times 10^6$ m, $G = 6{\cdot}7 \times 10^{-11}$ N m^2 kg^{-2}.

$$\therefore\ M = \frac{9{\cdot}8 \times (6{\cdot}4 \times 10^6)^2}{6{\cdot}7 \times 10^{-11}} = 6{\cdot}0 \times 10^{24}\ \text{kg}.$$

The volume of a sphere is $4\pi r^3/3$, where r is its radius. Thus the density, ρ, of the earth is approximately given by

$$\rho = \frac{M}{V} = \frac{gr^2}{4\pi r^3 G/3} = \frac{3g}{4\pi r G}.$$

By substituting known values of g, G, and r, the mean density of the earth is found to be about 5500 kg m^{-3}. The density may approach a value of 10000 kg m^{-3} towards the interior.

It is now believed that gravitational force travels with the speed of light. Thus if the gravitational force between the sun and earth were suddenly to disappear by the vanishing of the sun, it would take about 7 minutes for the effect to be experienced on the earth. The earth would then fly off along a tangent to its original curved path.

Gravitational and inertial mass

The mass m of an object appearing in the expression $F = ma$, force = mass × acceleration, is the *inertial mass*, as stated on p. 13. It is a measure of the reluctance of the object to move when forces act on it. It appears in $F = ma$ from Newton's second law of motion.

The 'mass' of the same object concerned in Newton's theory of gravitational attraction can be distinguished from the inertial mass. This is called the *gravitational mass*. If it is given the symbol m_g, then $F_g = GMm_g/r^2$, where F_g is the gravitational force, M is the mass of the earth and r its radius. Now $GM/r^2 = g$, the acceleration due to gravity (see above). Thus $F_g = m_g g = W$, the weight of the object.

In the simple pendulum theory on p. 48, we can derive the period T using W = weight = $m_g g$ in place of the symbols adopted there.

Thus
$$-m_g g\frac{y}{l} = ma,$$

or
$$a = -\frac{m_g g}{ml}\cdot y = -\omega^2 y.$$

$$\therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{m_g g}}.$$

Experiments show that to a high degree of accuracy, $T = 2\pi\sqrt{l/g}$ no matter what mass is used, that is, the period depends only on l and g. Thus $m = m_g$, or the gravitational mass is equal to the inertial mass to the best of our present knowledge.

Mass of Sun

The mass M_S of the sun can be found from the period of a satellite and its distance from the sun. Consider the case of the earth. Its period T is about 365 days or $365 \times 24 \times 3600$ seconds. Its distance r_S from the centre of the sun is about $1{\cdot}5 \times 10^{11}$ m. If the mass of the earth is m, then, for circular motion round the sun,

$$\frac{GM_S m}{r_S^2} = mr_S\omega^2 = \frac{mr_S 4\pi^2}{T^2},$$

$$\therefore M_S = \frac{4\pi^2 r_S^3}{GT^2} = \frac{4\pi^2 \times (1{\cdot}5 \times 10^{11})^3}{6{\cdot}7 \times 10^{-11} \times (365 \times 24 \times 3600)^2} = 2 \times 10^{30}\ \text{kg}.$$

Orbits round the earth

Satellites can be launched from the earth's surface to circle the earth. They are kept in their orbit by the gravitational attraction of the earth.

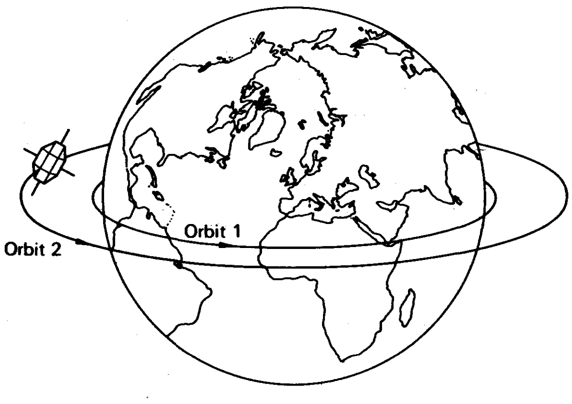

FIG. 2.22 Orbits round earth

Consider a satellite of mass m which just circles the earth of mass M

close to its surface in an orbit 1. Fig. 2.22 (i). Then, if r is the radius of the earth,

$$\frac{mv^2}{r} = G\frac{Mm}{r^2} = mg,$$

where g is the acceleration due to gravity at the earth's surface and v is the velocity of m in its orbit. Thus $v^2 = rg$, and hence, using $r = 6{\cdot}4 \times 10^6$ m and $g = 9{\cdot}8 \text{ m s}^{-2}$,

$$v = \sqrt{rg} = \sqrt{6{\cdot}4 \times 10^6 \times 9{\cdot}8} = 8 \times 10^3 \text{ m s}^{-1} \text{ (approx)},$$

$$= 8 \text{ km s}^{-1}.$$

The velocity v in the orbit is thus about 8 km s^{-1}. In practice, the satellite is carried by a rocket to the height of the orbit and then given an impulse, by firing jets, to deflect it in a direction parallel to the tangent of the orbit (see p. 66). Its velocity is boosted to 8 km s^{-1} so that it stays in the orbit. The period in orbit

$$= \frac{\text{circumference of earth}}{v} = \frac{2\pi \times 6{\cdot}4 \times 10^6 \text{ m}}{8 \times 10^3 \text{ m s}^{-1}}$$

$$= 5000 \text{ seconds (approx)} = 83 \text{ min}.$$

Parking Orbits

Consider now a satellite of mass m circling the earth in the plane of the equator in an orbit 2 concentric with the earth. Fig. 2.22 (ii). Suppose the direction of rotation as the same as the earth and the orbit is at a distance R from the centre of the earth. Then if v is the velocity in orbit,

$$\frac{mv^2}{R} = \frac{GMm}{R^2}.$$

But $GM = gr^2$, where r is the radius of the earth.

$$\therefore \frac{mv^2}{R} = \frac{mgr^2}{R^2}$$

$$\therefore v^2 = \frac{gr^2}{R}.$$

If T is the period of the satellite in its orbit, then $v = 2\pi R/T$.

$$\therefore \frac{4\pi^2 R^2}{T^2} = \frac{gr^2}{R}$$

$$\therefore T^2 = \frac{4\pi^2 R^3}{gr^2}. \quad . \quad . \quad . \quad \text{(i)}$$

If the period of the satellite in its orbit is exactly equal to the period of the earth as it turns about its axis, which is 24 hours, *the satellite will stay over the same place on the earth* while the earth rotates. This is sometimes called a 'parking orbit'. Relay satellites can be placed in parking orbits, so that television programmes can be transmitted continuously from one part of the world to another. *Syncom* was a satellite used for transmission of the Tokio Olympic Games in 1964.

Since $T = 24$ hours, the radius R can be found from (i). Thus from

$$R = \sqrt[3]{\frac{T^2gr^2}{4\pi^2}} \quad \text{and} \quad g = 9{\cdot}8\ \text{m s}^{-2}, r = 6{\cdot}4\times10^6\ \text{m},$$

$$\therefore R = \sqrt[3]{\frac{(24\times3600)^2\times9{\cdot}8\times(6{\cdot}4\times10^6)^2}{4\pi^2}} = 42400\ \text{km}$$

The height above the earth's surface of the parking orbit

$$= R-r = 42\,400-6\,400 = 36\,000\ \text{km}.$$

In the orbit, the velocity of the satellite

$$= \frac{2\pi R}{T} = \frac{2\pi\times42\,400}{24\times3600\ \text{seconds}} = 3{\cdot}1\ \text{km s}^{-1}.$$

Weightlessness

When a rocket is fired to launch a spacecraft and astronaut into orbit round the earth, the initial acceleration must be very high owing to the large initial thrust required. This acceleration, a, is of the order of $15g$, where g is the gravitational acceleration at the earth's surface.

Suppose S is the reaction of the couch to which the astronaut is initially strapped. Fig. 2.23 (i). Then, from $F = ma$, $S-mg = ma = m\,.\,15g$, where m is the mass of the astronaut. Thus $S = 16mg$. This force is 16 times the weight of the astronaut and thus, initially, he experiences a large force.

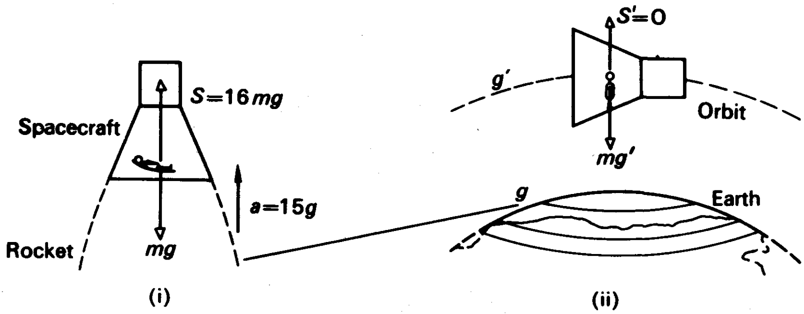

FIG. 2.23 Weight and weightlessness

In orbit, however, the state of affairs is different. This time the acceleration of the spacecraft and astronaut are both g' in magnitude, where g' is the acceleration due to gravity outside the spacecraft at the particular height of the orbit. Fig. 2.23 (ii). If S' is the reaction of the surface of the spacecraft in contact with the astronaut, then, for circular motion,

$$F = mg' - S' = ma = mg'.$$

Thus $S' = 0$. Consequently the astronaut becomes 'weightless'; he experiences no reaction at the floor when he walks about, for example. At the earth's surface we feel the reaction at the ground and are thus conscious of our weight. Inside a lift which is falling fast, the reaction at our feet diminishes. If the lift falls freely, the acceleration of objects inside is the same as that outside and hence the reaction on them is zero. This produces the sensation of 'weightlessness'. In orbit, as in Fig. 2.23 (ii), objects inside a spacecraft are also in 'free fall' because they have the same acceleration g' as the spacecraft. Consequently the sensation of weightlessness is experienced.

EXAMPLE

A satellite is to be put into orbit 500 km above the earth's surface. If its vertical velocity after launching is 2000 m s^{-1} at this height, calculate the magnitude and direction of the impulse required to put the satellite directly into orbit, if its mass is 50 kg. Assume $g = 10$ m s^{-2}; radius of earth, $R = 6400$ km.

Suppose u is the velocity required for orbit, radius r. Then, with usual notation,

$$\frac{mu^2}{r} = \frac{GmM}{r^2} = \frac{gR^2m}{r^2}, \text{ as } \frac{GM}{R^2} = g.$$

$$\therefore u^2 = \frac{gR^2}{r}.$$

Now $R = 6400$ km, $r = 6900$ km, $g = 10$ m s^{-2}.

$$\therefore u^2 = \frac{10 \times (6400 \times 10^3)^2}{6900 \times 10^3}.$$

$$\therefore u = 7700 \text{ m s}^{-1} \text{ (approx.)}.$$

At this height, vertical momentum

$U_Y = mv = 50 \times 2000 = 100\,000$ kg m s^{-1}.

Fig. 2.24.

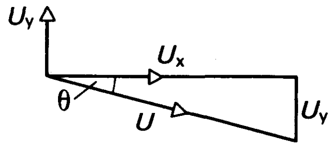

FIG. 2.24 Example

Horizontal momentum required $U_X = mu = 50 \times 7700 = 385\,000$ kg m s^{-1}.

$\therefore$ impulse needed, U, $= \sqrt{U_Y{}^2 + U_X{}^2} = \sqrt{100\,000^2 + 385\,000^2}$

$= 4{\cdot}0 \times 10^5$ kg m s^{-1} . . . . . (1)

Direction. The angle θ made by the total impulse with the horizontal or orbit tangent is given by $\tan\theta = U_Y/U_X = 100\,000/385\,000 = 0{\cdot}260$. Thus $\theta = 14{\cdot}6°$.

Magnitudes of acceleration due to gravity

(i) *Above the earth's surface.* Consider an object of mass m in an orbit of radius R from the centre, where $R > r$, the radius of the earth. Then, if g' is the acceleration due to gravity at this place,

$$mg' = \frac{GmM}{R^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

But, if g is the acceleration due to gravity at the earth's surface,

$$mg = \frac{GmM}{r^2} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Dividing (i) by (ii), $\therefore \frac{g'}{g} = \frac{r^2}{R^2}$, or $g' = \frac{r^2}{R^2} \cdot g$.

Thus above the earth's surface, the acceleration due to gravity g' varies *inversely as the square of the distance* from the centre. Fig. 2.25.

For a height h above the earth, $R = r+h$.

$$\therefore g' = \frac{r^2}{(r+h)^2} \cdot g = \frac{1}{\left(1+\frac{h}{r}\right)^2} \cdot g.$$

$$= \left(1+\frac{h}{r}\right)^{-2} \cdot g = \left(1-\frac{2h}{r}\right)g,$$

since powers of $(h/r)^2$ and higher can be neglected when h is small compared with r.

$$\therefore g-g' = \text{reduction in acceleration due to gravity.}$$

$$= \frac{2h}{r} \cdot g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(1)}$$

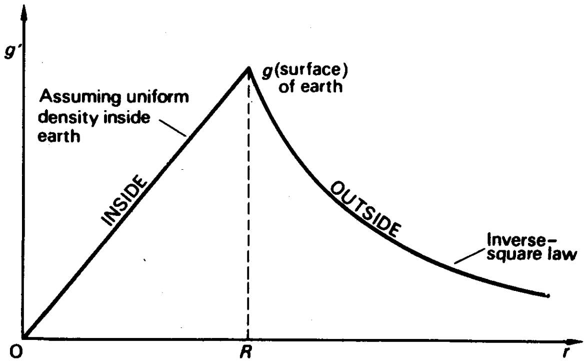

FIG. 2.25 Variation of g

(ii) *Below the earth's surface.* Consider an object of mass m at a point below the earth's surface. If its distance from the centre is b, the 'effective' mass M' of the earth which attracts it is that contained in a sphere of radius b. Assuming a constant density, then, since the mass of a sphere is proportional to *radius*^{3},

$$M' = \frac{b^3}{r^3}M,$$

where M is the mass of the earth. Suppose g'' is the acceleration due to gravity at the radius b. Then, from above,

$$mg'' = \frac{GmM'}{b^2} = \frac{GmMb}{r^2}.$$

Since $GM/r^2 = g$, it follows by substitution that

$$g'' = \frac{b}{r}g.$$

Thus assuming a uniform density of core, which is not the case in practice, the acceleration due to gravity g'' is directly proportional to the distance from the centre. Fig. 2.25.

If the depth below the earth's surface is h, then $b = r-h$.

$$\therefore\ g'' = \left(\frac{r-h}{r}\right)g = \left(1-\frac{h}{r}\right)g$$

$$\therefore\ g-g'' = \frac{h}{r}g \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Comparing (1) and (2), it can be seen that the acceleration at a distance h below the earth's surface is *greater* than at the same distance h above the earth's surface.

Potential

The *potential*, V, at a point due to the gravitational field of the earth is defined as numerically equal to the work done in taking a unit mass from infinity to that point. This is analogous to 'electric potential'. The potential at infinity is conventionally taken as *zero*.

For a point outside the earth, assumed spherical, we can imagine the whole mass M of the earth concentrated at its centre. The force of attraction on a unit mass outside the earth is thus GM/r^2, where r is the distance from the centre. The work done by the gravitational force in moving a distance δr towards the earth = force × distance = $GM.\delta r/r^2$. Hence the potential at a point distant a from the centre is given by

$$V_a = \int_{\infty}^{a} \frac{GM}{r^2}dr = -\frac{GM}{a} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

if the potential at infinity is taken as zero by convention. The negative sign indicates that the potential at infinity (zero) is *higher* than the potential close to the earth.

On the earth's surface, of radius r, we therefore obtain

$$V = -\frac{GM}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Velocity of Escape. Suppose a rocket of mass m is fired from the earth's surface Q so that it just escapes from the gravitational influence of the earth. Then work done = m × potential difference between infinity and Q.

$$= m \times \frac{GM}{r}.$$

$$\therefore \text{ kinetic energy of rocket} = \tfrac{1}{2}mv^2 = m \times \frac{GM}{r}.$$

$$\therefore\ v = \sqrt{\frac{2GM}{r}} = \text{velocity of escape.}$$

Now $$GM/r^2 = g.$$

$$\therefore v = \sqrt{2gr}.$$

$$\therefore v = \sqrt{2 \times 9{\cdot}8 \times 6{\cdot}4 \times 10^6} = 11 \times 10^3 \text{ m s}^{-1} = 11 \text{ km s}^{-1} \text{ (approx).}$$

With an initial velocity, then, of about 11 km s^{-1}, a rocket will completely escape from the gravitational attraction of the earth. It can be made to travel towards the moon, for example, so that eventually it comes under the gravitational attraction of this planet. At present, 'soft' landings on the moon have been made by firing retarding retro rockets.

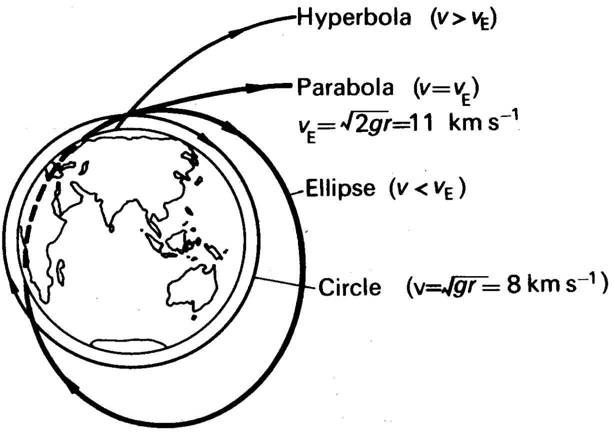

FIG. 2.26 Orbits

Summarising, with a velocity of about 8 km s^{-1}, a satellite can describe a circular orbit close to the earth's surface (p. 64). With a velocity greater than 8 km s^{-1} but less than 11 km s^{-1}, a satellite describes an elliptical orbit round the earth. Its maximum and minimum height in the orbit depends on its particular velocity. Fig. 2.26 illustrates the possible orbits of a satellite launched from the earth.

The molecules of air at normal temperatures and pressures have an average velocity of the order of 480 m s^{-1} or 0·48 km s^{-1} which is much less than the velocity of escape. Many molecules move with higher velocity than 0·48 km s^{-1} but gravitational attraction keeps the atmosphere round the earth. The gravitational attraction of the moon is much less than that of the earth and this accounts for the lack of atmosphere round the moon.

EXERCISES 2

(*Assume* $g = 10\ m\ s^{-2}$)

What are the missing words in the statements 1–6?

1. The force towards the centre in circular motion is called the . . . force.

2. In simple harmonic motion, the maximum kinetic energy occurs at the . . . of the oscillation.

3. The constant of gravitation G is related to g by . . .

4. In simple harmonic motion, the maximum potential energy occurs at the . . . of the oscillation.

5. Outside the earth, the acceleration due to gravity is proportional to . . . from the centre.

6. A satellite in orbit in an equatorial plane round the earth will stay at the same place above the earth if its period is . . . hours.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 7–10?

7. The earth retains its atmosphere because *A* the earth is spherical, *B* the velocity of escape is greater than the mean speed of molecules, *C* the constant of gravitation is a universal constant, *D* the velocity of escape is less than the mean speed of molecules, *E* gases are lighter than solids.

8. In simple harmonic motion, the moving object has *A* only kinetic energy, *B* mean kinetic energy greater than the mean potential energy, *C* total energy equal to the sum of the maximum kinetic energy and maximum potential energy, *D* mean kinetic energy equal to the mean potential energy, *E* minimum potential energy at the centre of oscillation.

9. If r is the radius of the earth and g is the acceleration at its surface, then the acceleration g' at an orbit distance R from the centre of the earth is given by *A* $g'/g = r/R$, *B* $g'/g = r^2/R^2$, *C* $g'/g = R^2/r^2$, *D* $g'/g = (R-r)^2/r^2$, *E* $g'/g = (R-r)/r$.

10. When water in a bucket is whirled fast overhead, the water does not fall out at the top of the motion because *A* the centripetal force on the water is greater than the weight of water, *B* the force on the water is opposite to gravity, *C* the reaction of the bucket on the water is zero, *D* the centripetal force on the water is less than the weight of water, *E* atmospheric pressure counteracts the weight.

Circular Motion

11. An object of mass 4 kg moves round a circle of radius 6 m with a constant speed of 12 m s^{-1}. Calculate (i) the angular velocity, (ii) the force towards the centre.

12. An object of mass 10 kg is whirled round a horizontal circle of radius 4 m by a revolving string inclined to the vertical. If the uniform speed of the object is 5 m s^{-1}, calculate (i) the tension in the string in kgf, (ii) the angle of inclination of the string to the vertical.

13. A racing-car of 1000 kg moves round a banked track at a constant speed of 108 km h^{-1}. Assuming the total reaction at the wheels is normal to the track, and the horizontal radius of the track is 100 m, calculate the angle of inclination of the track to the horizontal and the reaction at the wheels.

14. An object of mass 8·0 kg is whirled round in a vertical circle of radius 2m with a constant velocity of 6 m s^{-1}. Calculate the maximum and minimum tensions in the string.

15. Define the terms (*a*) *acceleration*, and (*b*) *force*. Show that the acceleration of a body moving in a circular path of radius r with uniform speed v is v^2/r, and draw a diagram to show the direction of the acceleration.

A small body of mass m is attached to one end of a light inelastic string of length l. The other end of the string is fixed. The string is initially held taut and horizontal, and the body is then released. Find the values of the following quantities when the string reaches the vertical position: (*a*) the kinetic energy of the body, (*b*) the velocity of the body, (*c*) the acceleration of the body, and (*d*) the tension in the string. (*O. & C.*)

16. Explain what is meant by *angular velocity*. Derive an expression for the force required to make a particle of mass m move in a circle of radius r with uniform angular velocity w.

A stone of mass 500 g is attached to a string of length 50 cm which will break if the tension in it exceeds 2·0 kgf. The stone is whirled in a vertical circle, the axis of rotation being at a height of 100 cm above the ground. The angular speed is very slowly increased until the string breaks. In what position is this break most likely to occur, and at what angular speed? Where will the stone hit the ground? (*C.*)

Simple Harmonic Motion

17. An object moving with simple harmonic motion has an amplitude of 2 cm and a frequency of 20 Hz. Calculate (i) the period of oscillation, (ii) the acceleration at the middle and end of an oscillation, (iii) the velocities at the corresponding instants.

18. Calculate the length in centimetres of a simple pendulum which has a period of 2 seconds. If the amplitude of swing is 2 cm, calculate the velocity and acceleration of the bob (i) at the end of a swing, (ii) at the middle, (iii) 1 cm from the centre of oscillation.

19. Define *simple harmonic motion*. An elastic string is extended 1 cm when a small weight is attached at the lower end. If the weight is pulled down $\frac{1}{4}$ cm and then released, show that it moves with simple harmonic motion, and find the period.

20. A uniform wooden rod floats upright in water with a length of 30 cm immersed. If the rod is depressed slightly and then released, prove that its motion is simple harmonic and calculate the period.

21. A simple pendulum, has a period of 4·2 seconds. When the pendulum is shortened by 1 m, the period is 3·7 seconds. From these measurements, calculate the acceleration due to gravity and the original length of the pendulum.

22. What is *simple harmonic motion*? Show how it is related to the uniform motion of a particle with velocity v in a circle of radius r.

A steel strip, clamped at one end, vibrates with a frequency of 50 Hz and an amplitude of 8 mm at the free end. Find (*a*) the velocity of the end when passing through the zero position, (*b*) the acceleration at the maximum displacement.

23. Explain what is meant by *simple harmonic motion*.

Show that the vertical oscillations of a mass suspended by a light helical spring are simple harmonic and describe an experiment with the spring to determine the acceleration due to gravity.

A small mass rests on a horizontal platform which vibrates vertically in simple harmonic motion with a period of 0·50 second. Find the maximum amplitude of the motion which will allow the mass to remain in contact with the platform throughout the motion. (*L.*)

24. *Define* simple harmonic motion and *state* a formula for its period. Show that under suitable conditions the motion of a simple pendulum is simple harmonic and hence obtain an expression for its period.

If a pendulum bob is suspended from an inaccessible point, by a string whose length may be varied, describe how to determine (*a*) the acceleration due to gravity, (*b*) the height of the point of suspension above the floor.

How and why does the value of the acceleration due to gravity at the poles differ from its value at the equator? (*L.*)

25. Derive an expression for the time period of vertical oscillations of small amplitude of a mass suspended from the free end of a light helical spring.

What deformation of the wire of the spring occurs when the mass moves? (*N.*)

26. Give *two* practical examples of oscillatory motion which approximate to simple harmonic motion. What conditions must be satisfied if the approximations are to be good ones.

A point mass moves with simple harmonic motion. Draw on the same axes sketch graphs to show the variation with position of (*a*) the potential energy, (*b*) the kinetic energy, and (*c*) the total energy of the particle.

A particle rests on a horizontal platform which is moving vertically in simple harmonic motion with an amplitude of 10 cm. Above a certain frequency, the thrust between the particle and the platform would become zero at some point in the motion. What is this frequency, and at what point in the motion does the thrust become zero at this frequency? (*C.*)

27. In what circumstances will a particle execute simple harmonic motion? Show how simple harmonic motion can be considered to be the projection on the diameter of a circle of the motion of a particle describing the circle with uniform speed.

The balance wheel of a watch vibrates with an angular amplitude of π radians and a period of 0·5 second. Calculate (*a*) the maximum angular speed, (*b*) the angular speed when the displacement is $\pi/2$, and (*c*) the angular acceleration when the displacement is $\pi/4$. If the radius of the wheel is r, calculate the maximum radial force acting on a small dust particle of mass m situated on the rim of the wheel. (*O. & C.*)

28. Prove that the bob of a simple pendulum may move with simple harmonic motion, and find an expression for its period.

Describe with full details how you would perform an experiment, based on the expression derived, to measure the value of the acceleration due to gravity. What factors would influence your choice of (*a*) the length of the pendulum, (*b*) the material of the bob, and (*c*) the number of swings to be timed? (*O. & C.*)

29. Define *simple harmonic motion* and show that the free oscillations of a simple pendulum are simple harmonic for small amplitudes.

Explain what is meant by damping of oscillations and describe an experiment to illustrate the effects of damping on the motion of a simple pendulum. Briefly discuss the difficulties you would encounter and indicate qualitatively the results you would expect to observe. (*O. & C.*)

30. What is meant by simple harmonic motion? Obtain an expression for the kinetic energy of a body of mass m, which is performing S.H.M. of amplitude a and period $2\pi/\omega$, when its displacement from the origin is x.

Describe an experiment, or experiments, to verify that a mass oscillating at the end of a helical spring moves with simple harmonic motion. (*C.*)

31. State the dynamical condition under which a particle will describe simple harmonic motion. Show that it is approximately fulfilled in the case of the bob of a simple pendulum, and derive, from first principles, an expression for the period of the pendulum.

Explain how it can be demonstrated from observations on simple pendulums, that the weight of a body at a given place is proportional to its mass. (*O. & C.*)

32. Define *simple harmonic motion.* Show that a heavy body supported by a light spiral spring executes simple harmonic motion when displaced vertically from its equilibrium position by an amount which does not exceed a certain value and then released. How would you determine experimentally the maximum amplitude for simple harmonic motion?

A spiral spring gives a displacement of 5 cm for a load of 500 g. Find the maximum displacement produced when a mass of 80 g is dropped from a height of 10 cm on to a light pan attached to the spring. (*N.*)

Gravitation

33. Calculate the force of attraction between two small objects of mass 5 and 8 kg respectively which are 10 cm apart. ($G = 6{\cdot}7 \times 10^{-11}$ N m^2 kg^{-2}.)

34. If the acceleration due to gravity is 9·8 m s^{-2} and the radius of the earth is 6400 km, calculate a value for the mass of the earth. ($G = 6{\cdot}7 \times 10^{-11}$ N m^2 kg^{-2}.) Give the theory.

35. Assuming that the mean density of the earth is 5500 kg m^{-3}, that the constant of gravitation is $6{\cdot}7 \times 10^{-11}$ N m^2 kg^{-2}, and that the radius of the earth is 6400 km, find a value for the acceleration due to gravity at the earth's surface. Derive the formula used.

36. How do you account for the sensation of 'weightlessness' experienced by the occupant of a space capsule (*a*) in a circular orbit round the earth, (*b*) in outer space? Give one other instance in which an object would be 'weightless'. (*N.*)

37. State Newton's law of universal gravitation. Distinguish between the gravitational constant (G) and the acceleration due to gravity (g) and show the relation between them.

Describe an experiment by which the value of g may be determined. Indicate the measurements taken and how to calculate the result. Derive any formula used. (*L.*)

38. State *Newton's law of gravitation.* What experimental evidence is there for the validity of this law?

A binary star consists of two dense spherical masses of 10^{30} kg and 2×10^{30} kg whose centres are 10^7 km apart and which rotate together with a uniform angular velocity ω about an axis which intersects the line joining their centres. Assuming that the only forces acting on the stars arise from their mutual gravitational attraction and that each mass may be taken to act at its centre, show that the axis of rotation passes through the centre of mass of the system and find the value of ω. ($G = 6{\cdot}7 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.) (*O. & C.*)

39. Assuming that the planets are moving in circular orbits, apply Kepler's laws to show that the acceleration of a planet is inversely proportional to the square of its distance from the sun. Explain the significance of this and show clearly how it leads to Newton's law of universal gravitation.

Obtain the value of g from the motion of the moon, assuming that its period of rotation round the earth is 27 days 8 hours and that the radius of its orbit is 60·1 times the radius of the earth. (Radius of earth = $6{\cdot}36 \times 10^6$ m.) (*N.*)

40. Explain what is meant by the *gravitation constant* (G), and describe an accurate laboratory method of measuring it. Give an outline of the theory of your method.

Assuming that the earth is a sphere of radius 6370 km and that $G = 6{\cdot}66 \times 10^{-11}$ N m^2 kg^{-2}, calculate the mean density of the earth. (*O. & C.*)

41. Assuming the earth to be perfectly spherical, give sketch graphs to show how (*a*) the acceleration due to gravity, (*b*) the gravitational potential due to the earth's mass, vary with distance from the surface of the earth for points external to it. If any other assumption has been made, state what it is.

Explain why, even if the earth were a perfect sphere, the period of oscillation of a simple pendulum at the poles would not be the same as at the equator. Still assuming the earth to be perfectly spherical, discuss whether the velocity required to project a body vertically upwards, so that it rises to a given height, depends on the position on the earth from which it is projected. (*C.*)

42. Explain what is meant by the *constant of gravitation*. Describe a laboratory experiment to determine it, showing how the result is obtained from the observations.

A proposed communication satellite would revolve round the earth in a circular orbit in the equatorial plane, at a height of 35880 km above the earth's surface. Find the period of revolution of the satellite in hours, and comment on the result. (Radius of earth = 6370 km, mass of earth = $5{\cdot}98 \times 10^{24}$ kg constant of gravitation = $6{\cdot}66 \times 10^{-11}$ N m^2 kg^{-2}.) (*N.*)